Wed, May 1, 2024

* HW 6 due on Friday
* Course Evals open
* O.H. may start up to 10 minutes late today

Topic 14 - Neighborhoods in Continuous Space

So far we've used a simple tweak in our continuous problems.

$x = (x_1, x_2, \ldots, x_d)$ d dimensions

$s = \text{tweak}(x) = x + (r_1\delta_1, r_2\delta_2, \ldots, r_d\delta_d)$

where each r_i is a uniform random # in $[-1, 1]$ and each δ_i is decided ahead of time.

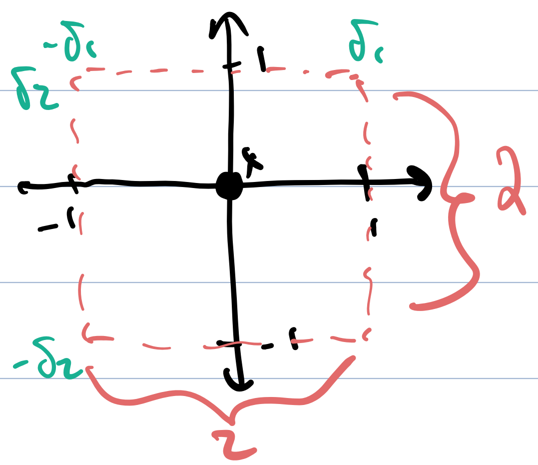
Works okay when d is small (2 or 3)

For now, assume $\delta_i = 1$ and $x = (0,0,0,\ldots,0)$ so

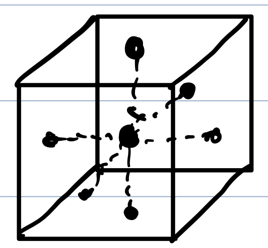
$$\text{tweak}(x) = (r_1, r_2, \ldots, r_d).$$

The new point tweak(x) is somewhere in the d-dimensional cube with side length 2 centered at the origin.

d=2



d=3



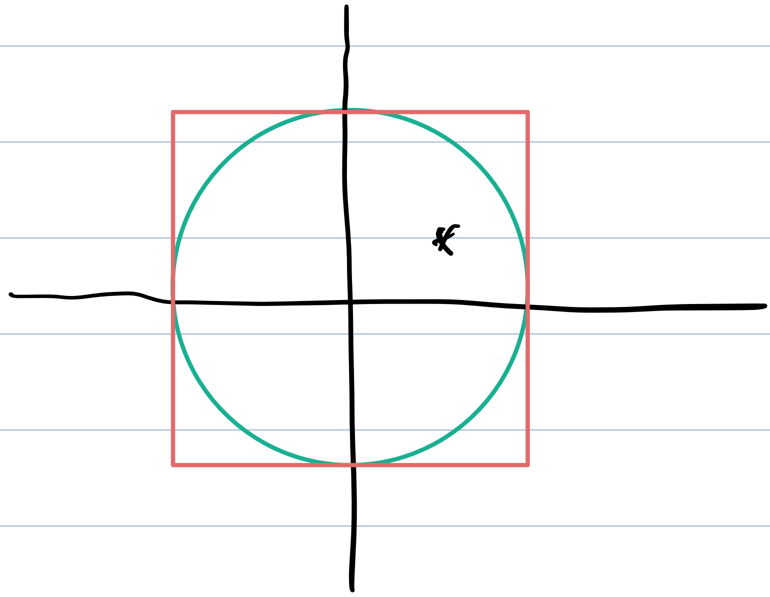
What is the farthest that s could be away from x?

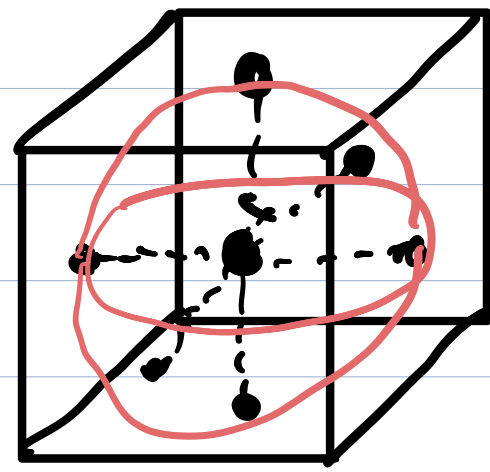
$d = 2 \rightsquigarrow \sqrt{2} \approx 1.41$

$d = 3 \qquad \sqrt{3} \approx 1.7$

In d-dimensional space $\rightsquigarrow \sqrt{d}$

Picking points in a cube can lead to very far away tweaks. Instead pick from a sphere.





* How do you pick points uniformly from the unit circle?

Idea 1) Rejection Sampling

Pick points in the square

Check if each point is in the circle

If so, keep it

If not, throw away and try again

Works, but very slow in higher dimensions

Idea 2) Pick $x \in [-1, 1]$

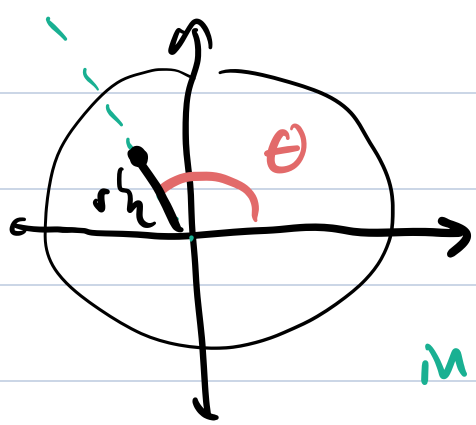
Pick $y \in [-\sqrt{1-x^2}, +\sqrt{1-x^2}]$

doesn't work

Idea 3) Pick an angle $\theta \in [0, 2\pi)$

and a radius $r \in [0, 1]$

doesn't work

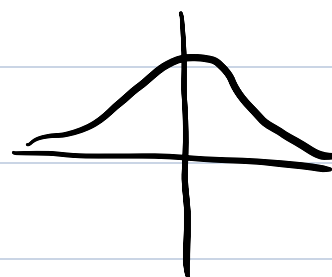


This one can be fixed in 2d by taking the square root of the radius

There is a way to pick uniformly from the d-dimensional sphere.

<u>Muller method:</u>

* Pick $(u_1, u_2, \ldots u_d)$ each from a Gaussian distribution (normal) with mean 0 and std. dev. 1



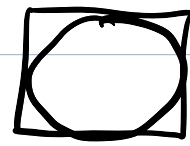
* Calculate the norm:

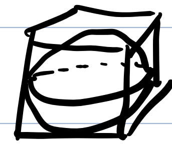
$$\sqrt{u_1^2 + u_2^2 + \ldots + u_d^2}$$

* Pick a random # $r \in [0, 1]$ uniformly

* Set $x = \dfrac{r^{1/d} \cdot \vec{u}}{\text{norm}}$ point in d-dim space

Difference between the d-sphere and the d-cube

2d: 

3d 

d	vol of the d-sphere	vol of the d-cube
1	2	2
2	3.14	4
3	4.19	8
4	4.93	16
5	5.26	32
6	5.168	64
10	2.55	$2^{10} = 1024$
20	0.03	$2^{20} = 1{,}048{,}576$

So, most of the volume of the cube is <u>not</u> in the sphere.

Muller method => nice small tweaks