Math 1450 - Calculus 1

Fri, Nov. 14

Announcements:

* Course withdrawal deadline: tonight
  come talk to me after class if you need your exam score today

* Next Thursday:
  → Quiz 9 (last one!)
  → Homework 12
  } 4.7 + 4.8

* Final Exam:
  Wednesday, Dec 8, 8pm - 10pm
  Weaster Auditorium

Today:

→ 4.7: L'hôpital's Rule

Office Hours

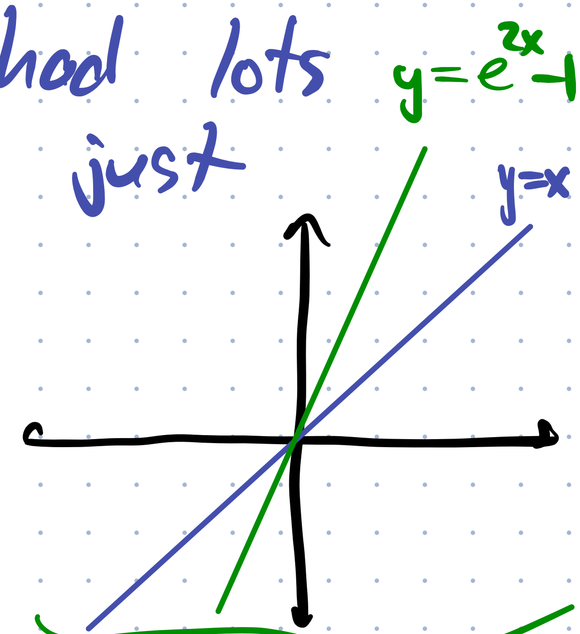
Mondays, 12-1

Wednesdays, 2-3

+ Help Desk! 12-1

Section 4.7 - L'Hôpital's Rule

Back when we learned limits, we had lots of examples that gave $\frac{0}{0}$ if you just plugged in the #!



zoomed in

Ex: $\lim_{x \to 0} \frac{e^{2x}-1}{x}$ $\qquad \frac{e^{0}-1}{0} = \frac{1-1}{0} = \frac{0}{0}$

When you zoom in to any function very closely at a point that's differentiable the function starts to look like a line.

the TL.

$$\underbrace{\lim_{x\to 0} \frac{e^{2x}-1}{x}}$$

the ratio of $e^{2x}-1$ vs. x, near $x=0$

$f(x) = e^{2x}-1$ $\quad$ $f'(x) = 2e^{2x}$

$g(x) = x$ $\quad$ $g'(x) = 1$

L.A. of $f(x)$ at $x=0$:

$$f(0) + f'(0)\cdot(x-0) = 2x$$

L.A. of $g(x)$ at $x=0$:

$$g(0) + g'(0)\cdot(x-0) = x$$

replace the top + bottom with their linear approximations

$$\lim_{x\to 0} \frac{2\cancel{x}}{\cancel{x}} = \lim_{x\to 0} 2 = \boxed{2}$$

General Rule for $\lim_{x \to a} \frac{f(x)}{g(x)}$ when $f(a) = 0$ and $g(a) = 0$

Linear approx. for $f(x)$ at $x=a$:

$$f(a) + f'(a)(x-a)$$

Linear approx. for $g(x)$ at $x=a$:

$$g(a) + g'(a) \cdot (x-a)$$

Near $x=a$:

$$\frac{f(x)}{g(x)} \approx \frac{\overset{0}{\cancel{f(a)}} + f'(a)(x-a)}{\underset{0}{\cancel{g(a)}} + g'(a) \cdot (x-a)} = \frac{f'(a)\cancel{(x-a)}}{g'(a)\cancel{(x-a)}} = \frac{f'(a)}{g'(a)}$$

Summary

if $f(a) = g(a) = 0$, then near $x = a$

$$\frac{f(x)}{g(x)} \approx \frac{f'(a)}{g'(a)}$$

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$ unless $g'(a) = 0$

L'Hôpital's Rule

If f and g are differentiable at $x=a$ and if $f(a)=0$ and $g(a)=0$ and $g'(a) \neq 0$, then:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Ex: $\lim_{x \to 0} \frac{\sin(x)}{x}$ Check $\frac{0}{0}$? ✓ $\sin(0) = 0$

$0 = 0$

LH: $\lim_{x \to 0} \frac{\sin(x)}{x} = \frac{\cos(0)}{1} = \frac{1}{1} = 1$ ✓

What if $\frac{f'(a)}{g'(a)}$ is still $\frac{0}{0}$? Do it again.

Ex: $\lim_{t \to 0} \frac{e^t - t - 1}{t^2} = \frac{1}{2}$

Plug in 0: $\frac{e^0 - 0 - 1}{0^2} = \frac{0}{0}$ ✓

$$\frac{(e^t - t - 1)'}{(t^2)'} = \frac{e^t - 1}{2t}$$

plug in 0: $\frac{e^0 - 1}{2 \cdot 0} = \frac{0}{0}$ ✓

$$\frac{(e^t - 1)'}{(2t)'} = \frac{e^t}{2}$$

plug in 0: $\frac{e^0}{2} = \boxed{\frac{1}{2}}$

Slightly more general version:

If f and g are differentiable and $f(a)=g(a)=0$, then:

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

$$\lim_{t\to 0}\frac{e^t-t-1}{t^2} = \lim_{t\to 0}\frac{e^t-1}{2t} = \lim_{t\to 0}\frac{e^t}{2} = \frac{1}{2}$$

L'Hopital's rule can be applied also when:

* if the limit is $x \to \infty$ or $x \to -\infty$ rather than $x \to a$
* if $\frac{f(a)}{g(a)}$ is $\frac{\infty}{\infty}$ $\frac{-\infty}{-\infty}$ $\frac{\infty}{-\infty}$ $\frac{-\infty}{\infty}$
* one-sided limits

Ex:

$$\lim_{x\to\infty} \frac{5x+e^{-x}}{7x}$$

as $x \to \infty$ $\longrightarrow \infty$

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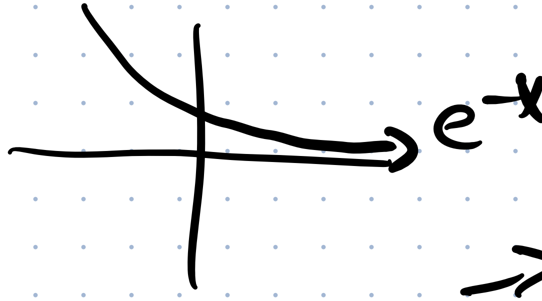


By LH:

$$= \lim_{x\to\infty} \frac{(5x+e^{-x})'}{(7x)'} = \lim_{x\to\infty} \frac{(5-e^{-x})}{7}$$

$$= \frac{5}{7}$$

$e^{-x} \to 0$

as $x \to \infty$

Sometimes you have to manipulate an expression to put it in a form where L'Hopital's Rule applies:

$x \to \infty$
$e^{-x} \to 0$

Ex: $\lim_{x \to \infty} (x \cdot e^{-x})$

$\infty \cdot 0$, not a L'Hopital form

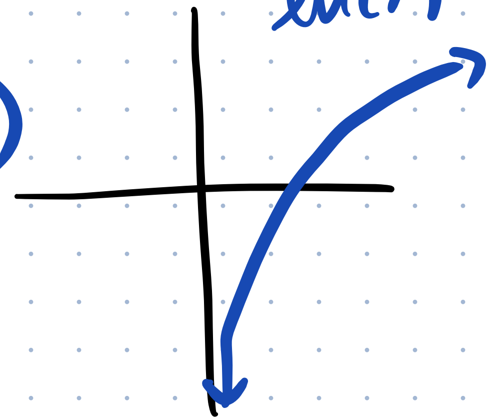
$x \cdot e^{-x} = \frac{x}{e^x}$ $\to \infty$, $\to \infty$

$= \lim_{x \to \infty} \frac{1}{e^x} = 0$

Ex:

$$\lim_{x\to 0^+} x\cdot \ln(x)$$

$0\cdot(-\infty)$



$$\frac{x}{1/\ln(x)}$$

$$= \lim_{x\to 0^+} \frac{\ln(x)}{1/x}$$

$$x^{-1} \rightsquigarrow -1\cdot x^{-2}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2}$$

$$\frac{1}{x}\cdot\left(-\frac{x^2}{1}\right)$$

$$= -x$$

$$= \lim_{x\to 0^+} (-x) = 0$$