Math 1450 - Calculus 1

Fri, Nov. 7

Announcements:

* HW 11 due Tuesday, covers 4.3+4.6, big sections!

* Exam 3 on Wednesday, Nov. 12 - 5pm-6pm
  covers 3.5, 3.6, 3.7, 3.9, 3.10
  4.1, 4.2, 4.3, 4.6

→ Exam 3 study guide is on our course website

Today:

→ 4.3: Optimization + Modeling

→ 4.6: Related Rates

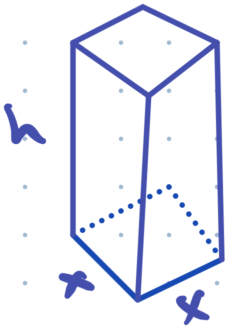
Office Hours

Mondays, 12-1

Wednesdays, 2-3

+ Help Desk! 12-1

Ex: What is the maximum volume of a closed box (all 6 sides) with a square base and exactly 24 in^2 surface area?



Volume: $x^2 \cdot h$

Surface area: $2x^2 + 4xh = 24$

constraint

$$\Rightarrow h = \frac{24 - 2x^2}{4x}$$

volume: $x^2 \cdot \left(\frac{24 - 2x^2}{4x}\right)$

maximize

Since this has 1 variable we use 4.2 methods to find global maximum.

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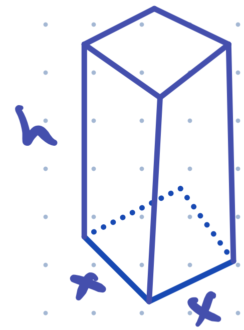


Volume: $x^2 \cdot h$

Surface area: $2 \; {}^{2} + \iota \qquad = 24$

Volume: $x^{\cancel{2}} \cdot \left(\dfrac{24 - 2x^2}{4\cancel{x}} \right) = \dfrac{1}{4}\left(24x - 2x^3\right)$

$= 6x - \frac{1}{2}x^3 = V(x)$

$\boxed{x > 0}$

$V'(x) = 6 - \frac{3}{2}x^2 = 0$

$\Rightarrow 6 = \frac{3}{2}x^2$

$\Rightarrow 12 = 3x^2$

$= 4 = x^2$

$\Rightarrow x = \pm 2$

only $x = 2$ is $\boxed{x > 0}$

no endpoints

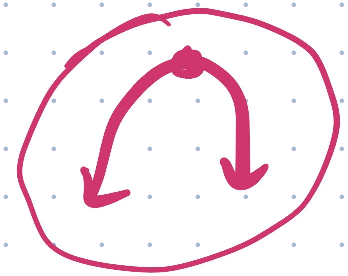
local max or min at $x = 2$?

$V''(x) = -3x$

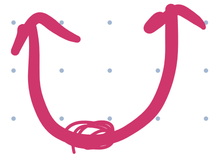
$V''(2) = -6$ (neg)

$CC\downarrow \Rightarrow x = 2$ is a local max

$f'' < 0 \Rightarrow f$ is cc$\downarrow$



$f'' > 0 \Rightarrow f$ is cc$\uparrow$

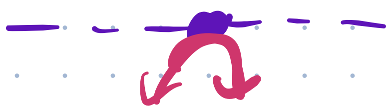


$f' = 0$

$f'' > 0$

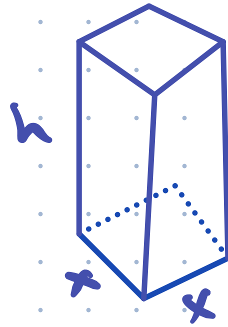
2nd deriv. test

$f' = 0$



$f'' < 0$

Ex: What is the maximum volume of a closed box (all 6 sides) with a square base and exactly 24 in^2 surface area?



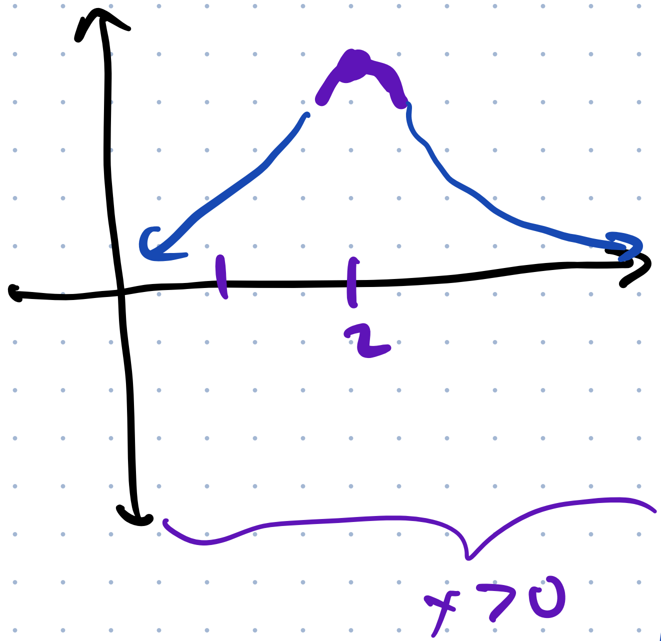
Volume: $x^2 \cdot h$ — Maximum vol = 8 in^3

Surface area: $2x^2 + 4xh = 24$

volume: $\cancel{x^2} \cdot \left(\frac{24 - 2x^2}{4\cancel{x}}\right) = \frac{1}{4}\left(24x - 2x^3\right)$

$= 6x - \frac{1}{2}x^3 = V(x)$

$\boxed{x > 0}$



Since there are no other critical points, this local max is a global max

$h = \frac{24 - 2x^2}{4x}$

$x = 2 \Rightarrow h = 2$

double check constraint: $2 \cdot (2)^2 + 4 \cdot 2 \cdot 2 = 24$ ✓

4.6: Related Rates

This section: Two different quantities are related.

* radius of a sphere vs. volume of a sphere
* position of a plane in air vs. the angle it's making with some landmark
* speed of a car vs. fuel efficiency

How is the rate of change of one related to the rate of change of the other?

Ex: A spherical snowball is melting. Its radius is decreasing at a constant rate of 2 cm/min from an initial radius of 70cm. How fast is the volume decreasing half an hour later? (No new concepts needed for this one.) $V'(30)$

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r of time t (minutes): $r(t) = 70 - 2t$

$$V = \frac{4}{3}\pi r^3 = \boxed{\frac{4}{3}\pi(70-2t)^3} = V(t)$$

$$V' = \frac{4}{\cancel{3}}\pi \cdot \cancel{3}(70-2t)^2 \cdot (-2) = -8\pi(70-2t)^2$$

$$V'(30) = -8\pi \cdot (10)^2 = -800\pi \approx \boxed{-2500\ \text{cm}^3/\text{min}}$$

What if we didn't have a formula $r(t)$ for the radius?

Ex: A spherical snowball is melting in such a way that at the instant the radius is 20cm, the radius is decreasing at a rate of 3cm/min. At what rate is the volume changing at the same instant?

$V = \frac{4}{3} \cdot \pi \cdot r^3$

Take the derivative of both sides with respect to t (time)

$V(t) = \frac{4}{3} \cdot \pi \left(r(t)\right)^3$

Ex: A spherical snowball is melting in such a way that at the instant the radius is 20cm, the radius is decreasing at a rate of 3cm/min. At what rate is the volume changing at the same instant?

$$V = \frac{4}{3} \cdot \pi \cdot r^3$$

$$V(t) = \frac{4}{3} \cdot \pi (r(t))^3$$

$$\frac{d}{dt}(V(t)) = \frac{d}{dt}\left(\frac{4}{3} \cdot \pi \cdot (r(t))^3\right)$$

$$V'(t) = \frac{4}{3} \cdot \pi \cdot \underbrace{\frac{d}{dt}\left(r(t)^3\right)}_{3 \cdot (r(t))^2 \cdot r'(t)}$$

$$\boxed{V'(t) = 4\pi (r(t))^2 \cdot r'(t)}$$

Tells us how the RoC of vol is related to RoC of radius

Ex: A spherical snowball is melting in such a way that at the instant the radius is $\boxed{20\text{cm}}$, the radius is decreasing at a rate of $\boxed{3\text{cm/min}}$. At what rate is the volume changing at the same instant?

$$V'(t) = 4\pi (r(t))^2 \cdot r'(t)$$

$$\frac{dV}{dt} = 4 \cdot \pi \cdot r^2 \cdot \frac{dr}{dt}$$

chain rule

Same thing in different notation

Want to know $\frac{dV}{dt}$ at the moment when

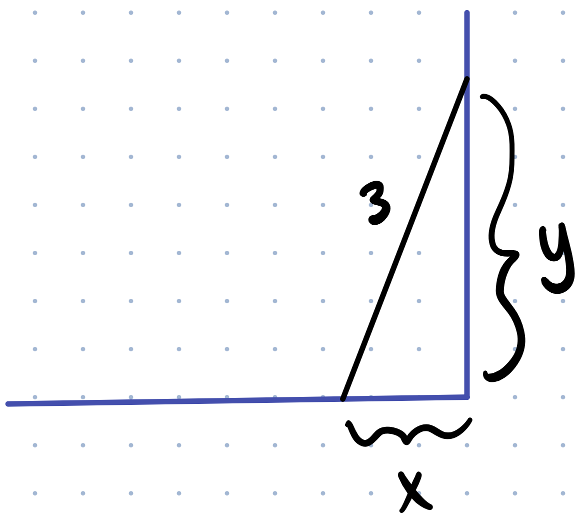
$r = 20$ and $\frac{dr}{dt} = -3$

$$\frac{dV}{dt} = 4 \cdot \pi \cdot 20^2 \cdot (-3) = \boxed{-4800\pi \ \frac{\text{cm}^3}{\text{min}}}$$

Ex: A 3-meter ladder stands against a high wall. The foot of the ladder moves outward at a constant speed of 0.1 m/s. When the foot is 1m from the wall, how fast is the top of the ladder falling? What about 2m?

(1) Draw a picture

(2) Give variable names to all relevant quantities!



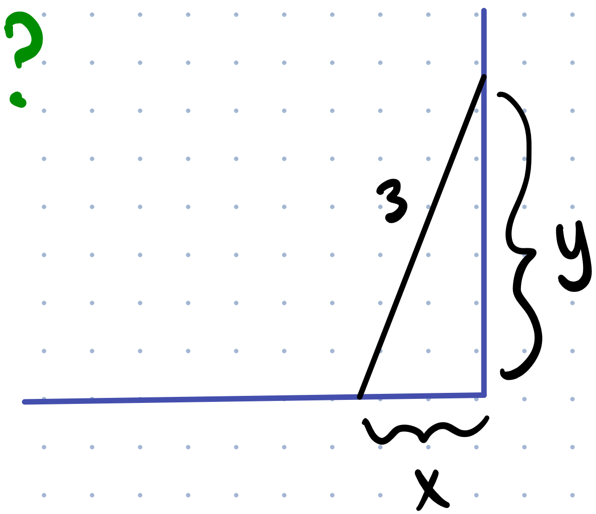
Q: when $x=1$ and $\frac{dx}{dt}=0.1$

what is $\frac{dy}{dt}$?

Need an equation that relates x and y

$$x^2+y^2=9$$

Ex: A 3-meter ladder stands against a high wall. The foot of the ladder moves outward at a constant speed of 0.1 m/s. When the foot is 1m from the wall, how fast is the top of the ladder falling? What about 2m?



$$x^2+y^2=9$$

$$\frac{d}{dt}(x^2+y^2)=\frac{d}{dt}(9)$$

$$2\cdot x\cdot\frac{dx}{dt}+2\cdot y\cdot\frac{dy}{dt}=0$$

chain rule

Know x, $\frac{dx}{dt}$

We can figure out y because $x^2+y^2=9$.

Solve for $\frac{dy}{dt}$