Monday, Nov 21 - Fall '22 (1)
Lecture #35

Announcements / Reminders

* Wiley Plus #12 due TUESDAY night (4.3)
* Wiley Plus #13 due the following Wednesday (4.6)

* Monday: Lecture
* Tuesday: Discussion
} Help Desk (12:30pm - 1:30pm) + Office Hours → on Microsoft Teams; Wiley Plus due!
* Wed - Fri: Thanksgiving Break

* ODS Email - today is the deadline

Section 4.6 - Related Rates

This section: two quantities that are related

* radius of a sphere vs. volume of a sphere
* position of a plane in flight vs. the angle it makes with a landmark on the ground.
* speed of a car vs. its fuel efficiency

As one changes over time, how does the other change?

Ex: A spherical snowball is melting. It's radius decreases at a constant rate of 2 cm/min from an initial value of 70 cm. How fast is the volume decreasing half an hour later?

No new concepts needed.

radius over time:

$$r = 70 - 2t \quad (r \text{ is in cm, } t \text{ is in minutes})$$

$$\text{volume} = \frac{4}{3}\pi \cdot (\text{radius})^3$$

$$V = \frac{4}{3}\pi(70-2t)^3$$

this is now a function that tells us the volume after t minutes

$$V'(t) = \frac{4}{3}\pi \cdot 3(70-2t)^2 \cdot (-2)$$

$$= -8\pi(70-2t)^2$$

$$V'(30) = -8\pi \cdot 100 = -800\pi \approx -2500 \ \frac{\text{cm}^3}{\text{min}}$$

What if we didn't have a nice function for the radius at every point in time? $70-2t$

Ex: A spherical snowball is melting in such a way that at the instant that its radius is 20 cm, the radius is decreasing at a rate of 3 cm/min. At what rate is the volume changing at that instant?

We know r and r' at one instant, and want to figure V' at the same instant.

Step 1: Write down a formula that relates the two quantities. (radius and volume)

$$V = \frac{4}{3}\pi r^3$$

Step 2: Take the derivative of both sides with respect to a new variable t.

$$V(t) = \frac{4}{3}\pi (r(t))^3 \qquad \left(\begin{array}{l} \text{"}V\text{"} \to \text{"}V(t)\text{"} \\ \text{"}r\text{"} \to \text{"}r(t)\text{"} \end{array} \right)$$

$$\frac{d}{dt}\left(V(t)\right) = \frac{d}{dt}\left(\frac{4}{3}\pi (r(t))^3\right)$$

$$V'(t) = \frac{4}{3}\pi \cdot 3(r(t))^2 \cdot r'(t)$$

$$V'(t) = 4\pi(r(t))^2 \cdot r'(t)$$

$$\frac{dV}{dt} = 4\pi r^2 \cdot \frac{dr}{dt}$$ ← same thing in two different notations

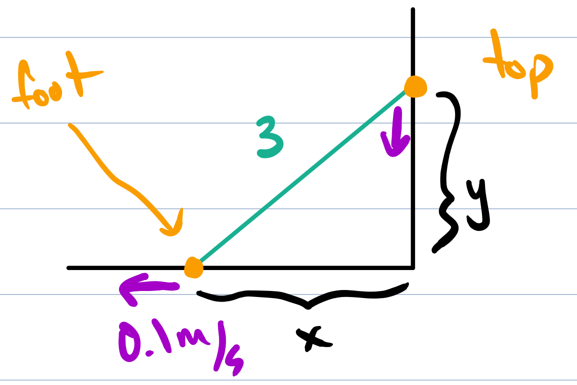
Step 3: Plug stuff into this to get your answer.

Question: $r = 20$
$r' = -3$
What is V'?

$$\frac{dV}{dt} = 4 \cdot \pi \cdot (20)^2 \cdot (-3) = -4800\pi \ \frac{cm^3}{min}$$

Mental Check: Does the sign (+ or −) make sense?

Ex A 3-meter ladder stands against a wall. The foot of the ladder moves outward at a constant speed of 0.1 m/s. When the foot is 1m from the wall, how fast is the top of the ladder falling? What about when the foot is 2m from the wall?



Know: $x = 1\text{m}$

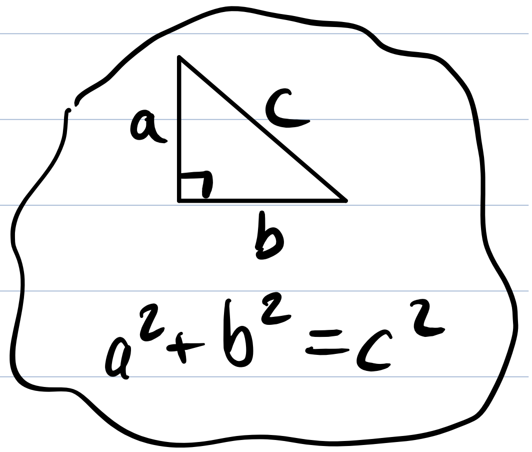
$$\frac{dx}{dt} = 0.1 \text{ m/s}$$

What is $\frac{dy}{dt}$? (should be negative)

Step 1) Some formula relating x and y:

$$x^2 + y^2 = 3^2$$

* don't need to solve for one variable *



Step 2) "x" → "x(t)"
"y" → "y(t)"

$$(x(t))^2 + (y(t))^2 = 9$$

Derivative of both sides w.r.t. t

$$\frac{d}{dt}\left((x(t))^2 + (y(t))^2\right) = \frac{d}{dt}(9)$$

$$2\cdot x(t)\cdot x'(t) + 2\cdot y(t)\cdot y'(t) = 0$$

Sometimes written:

$$2xx' + 2yy' = 0$$

OR:

$$2\cdot x\cdot \frac{dx}{dt} + 2\cdot y\cdot \frac{dy}{dt} = 0$$

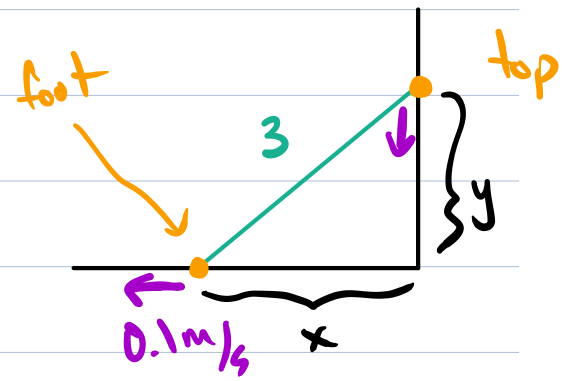
Formula that relates x, y, x', and y'!

Step 3) We know:

$x = 1$

$x' = 0.1$

$y = ?$ $\sqrt{8}$



Want y'

Often we'll need to first use the initial equation $(x^2 + y^2 = 9)$ to solve.

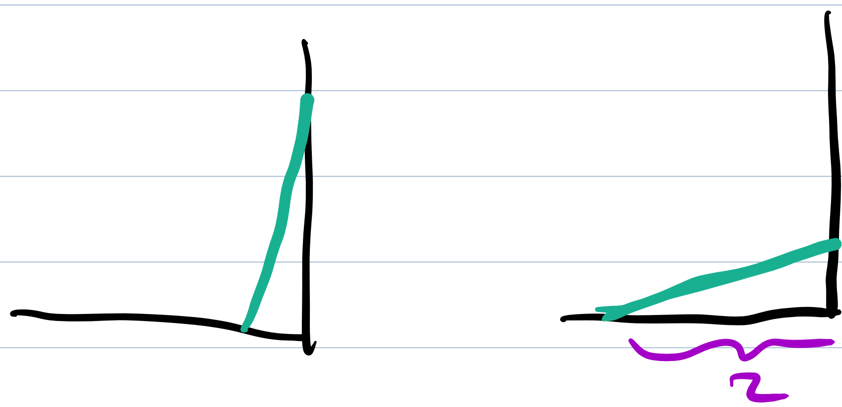
$$x = 1 \Rightarrow 1^2 + y^2 = 9 \Rightarrow y = \sqrt{8}$$

$$2 \cdot x \cdot \frac{dx}{dt} + 2 \cdot y \cdot \frac{dy}{dt} = 0$$

$$\Rightarrow \quad 2 \cdot (1) \cdot (0.1) + 2 \cdot \sqrt{8} \cdot \left(\frac{dy}{dt}\right) = 0$$

$$\Rightarrow \quad 2 \cdot \sqrt{8} \cdot \left(\frac{dy}{dt}\right) = -0.2$$

$$\Rightarrow \quad \frac{dy}{dt} = -\frac{0.2}{2\sqrt{8}} \approx -0.035 \text{ m/s}$$



What about when $x = 2$?

$$x^2 + y^2 = 9 \Rightarrow 4 + y^2 = 9 \Rightarrow y = \sqrt{5}$$

$$2 \cdot x \cdot \frac{dx}{dt} + 2 \cdot y \cdot \frac{dy}{dt} = 0$$

$$2 \cdot 2 \cdot (0.1) + 2 \cdot \sqrt{5} \cdot \left(\frac{dy}{dt}\right) = 0$$

$$\Rightarrow \quad \frac{dy}{dt} = -0.089 \text{ m/s}$$