The Rearrangement Conjecture

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The Generalized Factor Order

Given a poset P and words $u = u_1 \dots u_n$ and $v = v_1 \dots v_n$ over P, we say that v *dominates* u if, for all i,

$$v_i \geq_P u_i$$

Then, u is a *generalized factor* of w if there exist words $w^{(1)}$ and $w^{(2)}$ such that

$$w = w^{(1)} v w^{(2)}$$

for some word v which dominates u. This is denoted

$$u \leq_{\text{gfo}} w.$$

▷ **Example:** Over the poset $\mathbb{P}$ of positive integers, $3123 \leq_{\text{gfo}} 1\mathbf{4233}14$.

For everything below, $P = \mathbb{P}$.

Wilf-Equivalence of Generalized Factors

Given a word $w \in \mathbb{P}^*$, let $|w|$ be the length of w and let $\|w\|$ be the sum of the letters of w.

▷ **Example:** If $w = 1423314$, then $|w| = 7$ and $\|w\| = 18$.

We are primarily interested in the generating function which counts the set of all words in $\mathbb{P}^*$ according to their length, the sum of their entries, and the number of factors dominating a given word u that they contain:

$$A_u(x, y, z) = \sum_{w \in \mathbb{P}^*} x^{|w|} y^{\|w\|} z^{\#\text{ of factors dominating } u}$$

Two words u and v are said to be *Wilf-equivalent* (denoted $u \sim v$) if

$$A_u(x, y, 0) = A_v(x, y, 0).$$

$A_u(x, y, 0)$ enumerates the words in $\mathbb{P}^*$ that *avoid* the factor u by length and sum.

▷ **Example:** $A_{122}(x, y, 0) = 1 + xy + (x + x^2)y^2 + (x + 2x^2 + x^3)y^3 + \cdots$.

The Rearrangement Conjecture

The Rearrangement Conjecture (Kitaev, Liese, Remmel, Sagan) If two words in $\mathbb{P}^*$ are Wilf-equivalent, then they are rearrangements of each other.

The converse is **false**. By constructing automata, we find

$$A_{122}(x, y, 0) = \frac{1 - 2y + (1 + x)y^2 - xy^3 + x^2y^4}{1 - (2 + x)y + (1 + 2x)y^2 - (x + x^2)y^3 + x^2y^4},$$

while

$$A_{212}(x, y, 0) = \frac{1 - 2y + (1 + x)y^2 - (x - x^2)y^3 + x^3y^5}{(1 - y + x^2y^3)(1 - (1 + x)y + xy^2 - x^2y^3)}.$$

In particular, $[x^4y^7]A_{122}(x, y, 0) = 13$, while $[x^4y^7]A_{212}(x, y, 0) = 12$.

Strong Wilf-Equivalence

We consider a more restrictive version of Wilf-equivalence.

We say that u and v are *strongly Wilf-equivalent* if

$$A_u(x, y, z) = A_v(x, y, z).$$

We prove that if two words in $\mathbb{P}^*$ are strongly Wilf-equivalent, then they are rearrangements of each other.

Conjecture: Wilf-equivalence and strong Wilf-equivalence are equivalent conditions.

The Cluster Method

An *m-cluster* of u is a word $c \in \mathbb{P}^*$ which consists entirely of m *marked* overlapping occurrences of the factor u.

Every letter of the word must be contained in at least one marked occurrence of u.

The same word may have different possible markings. For example, the word 3423523 can be both a 2-cluster and a 3-cluster of 3123.

$$3\ 4\ 2\ 3\ 5\ 2\ 3 \qquad\qquad 3\ 4\ 2\ 3\ 5\ 2\ 3$$

The *cluster generating function* is defined by

$$C_u(x, y, z) = \sum_{m \geq 1} z^m \sum_{\substack{m\text{-clusters}\\ c \text{ of } u}} x^{|c|} y^{\|c\|}.$$

The *sieve method* (basically, inclusion-exclusion) shows that $C_u(x, y, z)$ and $A_u(x, y, z)$ are related by

$$A_u(x, y, z) = \frac{1}{1 - \frac{xy}{1-y} - C_u(x, y, z - 1)}.$$

Simplifying even further, we call a cluster *minimal* if none of its entries can be decreased without destroying a marked factor. The minimal clusters are given by the generating function

$$M_u(x, y, z) = \sum_{m \geq 1} z^m \sum_{\substack{\text{minimal}\\ m\text{-clusters}\\ c \text{ of } u}} x^{|c|} y^{\|c\|}.$$

It is easy to see that

$$C_u(x, y, z) = M_u\left(\frac{x}{1-y}, y, z\right).$$

This all boils down to:

$$\boxed{u \text{ and } v \text{ are strongly Wilf-equivalent if and only if } M_u(x, y, z) = M_v(x, y, z).}$$

Working with Minimal Clusters

Minimal clusters can be constructed by overlapping occurrences of a factor and taking the maximum of each column.

$$\begin{array}{cccccccccc}
3 & 1 & 2 & 5 & & & & & & \\
 & 3 & 1 & 2 & 5 & & & & & \\
 & & & 3 & 1 & 2 & 5 & & & \\
 & & & & & & 3 & 1 & 2 & 5 \\
\hline
3 & 3 & 2 & 5 & 5 & 2 & 5 & 1 & 2 & 5
\end{array}$$

A minimal 4-cluster of 3125.

▷ **Example:** We can find the first couple of terms of $M_{132}(x, y, z)$ by considering possible overlaps.

$\boldsymbol{x^3y^6z^1}$

$$\begin{array}{ccc}
1 & 3 & 2 \\
\hline
1 & 3 & 2
\end{array}$$

$\boldsymbol{x^4y^9z^2}$

$$\begin{array}{cccc}
1 & 3 & 2 & \\
 & 1 & 3 & 2 \\
\hline
1 & 3 & 3 & 2
\end{array}$$

$\boldsymbol{x^5y^{11}z^2}$

$$\begin{array}{ccccc}
1 & 3 & 2 & & \\
 & & 1 & 3 & 2 \\
\hline
1 & 3 & 2 & 3 & 2
\end{array}$$

$\boldsymbol{x^5y^{12}z^3}$

$$\begin{array}{ccccc}
1 & 3 & 2 & & \\
 & 1 & 3 & 2 & \\
 & & 1 & 3 & 2 \\
\hline
1 & 3 & 3 & 3 & 2
\end{array}$$

$\boldsymbol{x^6y^{14}z^3}$

$$\begin{array}{cccccc}
1 & 3 & 2 & & & \\
 & 1 & 3 & 2 & & \\
 & & & 1 & 3 & 2 \\
\hline
1 & 3 & 3 & 2 & 3 & 2
\end{array}$$

$\boldsymbol{x^6y^{14}z^3}$

$$\begin{array}{cccccc}
1 & 3 & 2 & & & \\
 & & 1 & 3 & 2 & \\
 & & & 1 & 3 & 2 \\
\hline
1 & 3 & 2 & 3 & 3 & 2
\end{array}$$

$\boldsymbol{x^7y^{16}z^3}$

$$\begin{array}{ccccccc}
1 & 3 & 2 & & & & \\
 & & 1 & 3 & 2 & & \\
 & & & & 1 & 3 & 2 \\
\hline
1 & 3 & 2 & 3 & 2 & 3 & 2
\end{array}$$

$\boldsymbol{x^6y^{15}z^4}$

$$\begin{array}{cccccc}
1 & 3 & 2 & & & \\
 & 1 & 3 & 2 & & \\
 & & 1 & 3 & 2 & \\
 & & & 1 & 3 & 2 \\
\hline
1 & 3 & 3 & 3 & 3 & 2
\end{array}$$

Recovering the Multiset of Values of u from $M_u(x, y, z)$

To prove that strongly Wilf-equivalent words u and v must be rearrangements of each other, we prove that from $M_u(x, y, z)$ one can recover the multiset of entries of u.

We demonstrate the idea by proving the case of a word $u = u_1u_2u_3u_4$ of length 4.

Let $u_I = \max\{u_i : i \in I\}$ for $I \subseteq \{1, 2, 3, 4\}$. Consider the minimal 2-, 3-, and 4-clusters. The minimal 2-clusters are shown below.

$$\begin{array}{ccccc}
u_1 & u_2 & u_3 & u_4 & \\
 & u_1 & u_2 & u_3 & u_4 \\
\hline
u_1 & u_{1,2} & u_{2,3} & u_{3,4} & u_4
\end{array}
\qquad
\begin{array}{cccccc}
u_1 & u_2 & u_3 & u_4 & & \\
 & & u_1 & u_2 & u_3 & u_4 \\
\hline
u_1 & u_2 & u_{1,3} & u_{2,4} & u_3 & u_4
\end{array}
\qquad
\begin{array}{ccccccc}
u_1 & u_2 & u_3 & u_4 & & & \\
 & & & u_1 & u_2 & u_3 & u_4 \\
\hline
u_1 & u_2 & u_3 & u_{1,4} & u_2 & u_3 & u_4
\end{array}$$

Therefore,

$$[z^2]M_u = x^5y^{u_1+u_{1,2}+u_{2,3}+u_{3,4}+u_4} + x^6y^{u_1+u_2+u_{1,3}+u_{2,4}+u_3+u_4} + x^7y^{u_1+u_2+u_3+u_{1,4}+u_2+u_3+u_4}.$$

and so, for example,

$$\frac{d}{dy}\left(\left([x^7z^2] - [x^6z^2]\right) M_u\right)\bigg|_{y=1} = u_2 + u_3 + u_{1,4} - u_{1,3} - u_{2,4}.$$

We can summarize the information about 2-clusters by counting the number of appearances of each maximum in each length.

length	u_1	u_2	u_3	u_4	$u_{1,2}$	$u_{1,3}$	$u_{1,4}$	$u_{2,3}$	$u_{2,4}$	$u_{3,4}$
5	1			1	1			1		1
6	1	1	1	1		1			1	
7	1	2	2	1			1			

It turns out that the 3- and 4-clusters are more helpful. The 3-clusters are:

length	u_1	u_2	u_3	u_4	$u_{1,2}$	$u_{1,3}$	$u_{1,4}$	$u_{2,3}$	$u_{2,4}$	$u_{3,4}$	$u_{1,2,3}$	$u_{1,2,4}$	$u_{1,3,4}$	$u_{2,3,4}$	$u_{1,2,3,4}$
6	1			1	1					1	1			1	
7	2	1	1	2	1	1		2	1	1		1	1		
8	3	3	3	3	2	2	2	2	2	2					
9	2	4	4	2		2	2		2						
10	1	3	3	1			2								

The 4-clusters are:

length	u_1	u_2	u_3	u_4	$u_{1,2}$	$u_{1,3}$	$u_{1,4}$	$u_{2,3}$	$u_{2,4}$	$u_{3,4}$	$u_{1,2,3}$	$u_{1,2,4}$	$u_{1,3,4}$	$u_{2,3,4}$	$u_{1,2,3,4}$
7	1			1	1					1	1			1	1
8	3	1	1	3	2	1		2	1	2	2	2	2	2	
9	6	4	4	6	5	4	3	5	4	5	2	2	2	2	
10	7	9	9	7	4	7	6	6	7	4		2	2		
11	6	12	12	6	3	6	9	3	6	3					
12	3	9	9	3		3	6		3						
13	1	4	4	1			3								

Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ be the four values of entries in u. By cleverly combining highlighted rows in the tables above, we can find:

$$\frac{d}{dy}\left(\left([x^7z^4] - [x^6z^3]\right) M_u\right)\bigg|_{y=1} = u_{1,2,3,4} = \lambda_1$$

$$\frac{d}{dy}\left(\left([x^8z^4] - [x^7z^3] - [x^6z^3]\right) M_u\right)\bigg|_{y=1} = u_{1,2,3} + u_{1,2,4} + u_{1,3,4} + u_{2,3,4} = 3\lambda_1 + \lambda_2$$

$$\frac{d}{dy}\left(\left([x^9z^4] - [x^8z^3] - [x^7z^3] - [x^6z^3]\right) M_u\right)\bigg|_{y=1} = \cdots = 6\lambda_1 + 3\lambda_2 + \lambda_3$$

Additionally, the smallest exponent of y in M_u equals $\|u\| = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4$. From these four pieces of data, we recover the four values of entries in u, proving the case of length 4.

Open Questions

Proving that strong Wilf-equivalence is equivalent to Wilf-equivalence would prove the full Rearrangement Conjecture.

We have verified this conjecture computationally for factors of weight up to 11 contained in words of weight up to 20.

▷ **Question:** Which rearrangements are Wilf-equivalent?

For example, we have proved that if $x, y \leq a, b, c$, then $axbyc \sim aybxc$.