Pattern-Avoiding Involutions: Exact and Asymptotic Enumeration

(joint work with Miklós Bóna, Cheyne Homberger, and Vince Vatter)

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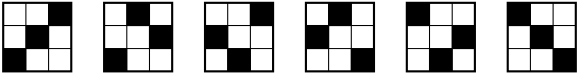

Permutation Patterns 2014

July 7, 2014

Growth Rates of Permutation Classes

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To this end, we define the upper and lower growth rates of a class $\mathcal{C}$ as

$$\overline{\mathrm{gr}}(\mathcal{C}) = \limsup_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|} \qquad \text{and} \qquad \underline{\mathrm{gr}}(\mathcal{C}) = \liminf_{n\to\infty} \sqrt[n]{|\mathcal{C}_n|}.$$

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In this context, the Marcos-Tardos Theorem says that $\underline{\text{gr}}(\mathcal{C})$ and $\overline{\text{gr}}(\mathcal{C})$ are finite.

Growth Rates of Permutation Classes

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Arratia proved that *principally based* classes (those of the form $\mathrm{Av}(\pi)$) do have proper growth rates.

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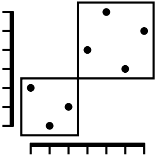

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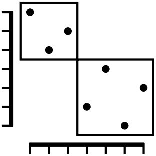

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We can now present Arratia's proof.

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The sum operation defines an injection

$$\oplus : \mathrm{Av}_m(\beta) \times \mathrm{Av}_n(\beta) \to \mathrm{Av}_{m+n}(\beta),$$

from which it follows that $|\mathrm{Av}_{m+n}(\beta)| \geq |\mathrm{Av}_m(\beta)||\mathrm{Av}_n(\beta)|$.

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Fekete's Lemma then implies that the growth rate exists (and the Marcos-Tardos Theorem implies that the growth rate is finite). □

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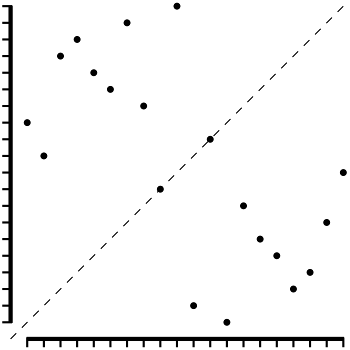

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We will call a set of the form $\text{Av}^I(\pi)$ principally based, although this is a bit disingenuous: when $\pi \neq \pi^{-1}$, we have

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- There are 8 different enumerations of $\mathrm{Av}^I(\beta)$ for $|\beta| = 4$

Enumerations (as presented by Jaggard)

	1324	**1234**	**4231**	**2431**	**1342**	**2341**	**3421**	**2413**
$\lvert \mathrm{Av}_5^I(\beta) \rvert$	21	21	21	24	24	25	25	24
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Is $\mathrm{Av}^I(1324)$ really the smallest?

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(And, the growth rate of $\text{Av}^I(2413)$ is known to be ≈ 3.15, so unless $\text{gr}(\text{Av}(1324)) < 3.15^2 < 9.93$, $\text{Av}_n^I(1324)$ should also surpass $\text{Av}_n^I(2413)$.)

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$\lvert Av_{12}^I(\beta)\rvert$	16238	18477	18322	15511	15272	22878	16716	27246
$\lvert Av_{13}^I(\beta)\rvert$	40914	46825	47560	41835	40758	60794	46246	77132
$\lvert Av_{14}^I(\beta)\rvert$	103954	118917	124358	113634	112280	161668	128414	221336
$\lvert Av_{15}^I(\beta)\rvert$	262298	301734	323708	310572	304471	429752	361493	635078
$\lvert Av_{16}^I(\beta)\rvert$	665478	766525	846766	853467	852164	1142758	1020506	1839000
$\lvert Av_{17}^I(\beta)\rvert$	1680726	1946293	2208032	2356779	2341980	3038173	2913060	5331274
$\lvert Av_{18}^I(\beta)\rvert$	4260262	4944614	5777330	6536382	6640755	8078606	8335405	15555586
$\lvert Av_{19}^I(\beta)\rvert$	10766470	12557685	15082372	18199284	18460066	21479469	24067930	45465412
$\lvert Av_{20}^I(\beta)\rvert$	27274444	31900554	39469786	50852019	52915999	57113888	69646035	133517130

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$\lvert \mathrm{Av}_{20}^I(\beta)\rvert$	27274444	31900554	39469786	50852019	52915999	57113888	69646035	133517130
growth rate	?	≈ 2.54	≈ 2.62	3	$(3.13, 4.84)$	?	?	≈ 3.15

1342- AND 2341-AVOIDING INVOLUTIONS

We find the generating functions for $\text{Av}^I(1342)$ and $\text{Av}^I(2341)$ by using the fact that for any set $\mathcal{S}$ of permutations:

1342- and 2341-Avoiding Involutions

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As usual, the hard part is counting the simple permutations in the set and their allowed inflations.

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It turns out that the simple involutions of $\mathrm{Av}^I(1342)$ coincide exactly with the simple involutions of $\mathrm{Av}^I(123)$.

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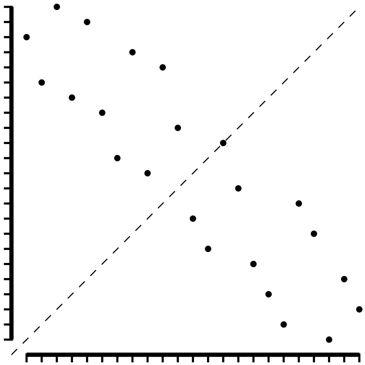

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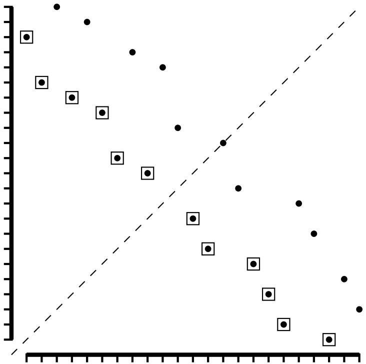

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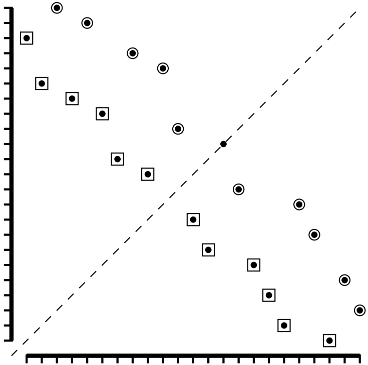

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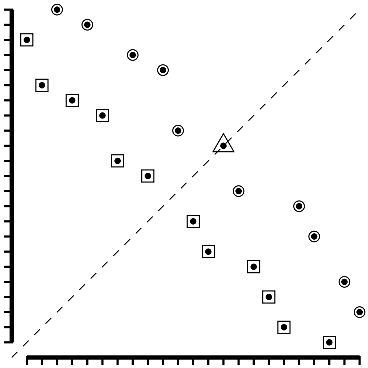

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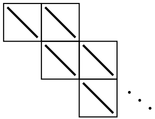

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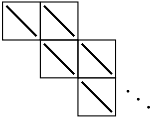

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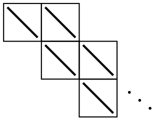

To assure that every permutation has a unique way of being constructed, we require that the first box contains the longest possible initial decreasing sequence.

Every subsequent box contains either:

- every entry above and to the right of existing entries, or
- every entry below and to the left of existing entries.

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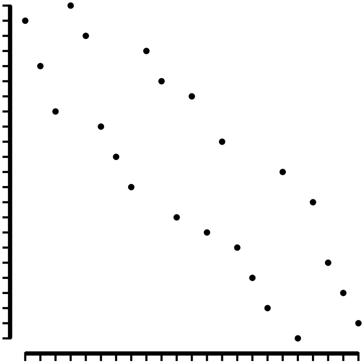

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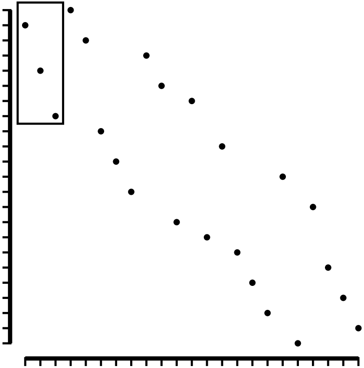

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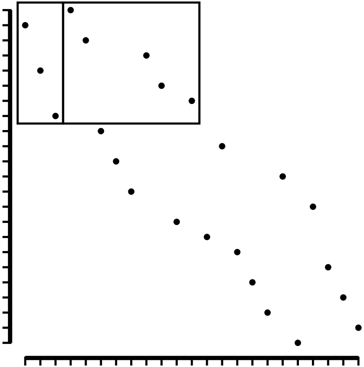

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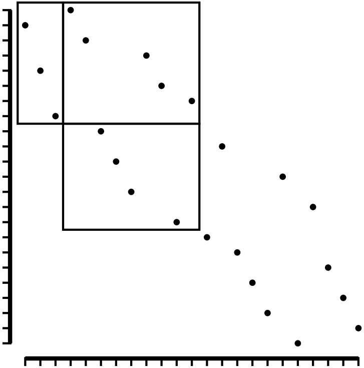

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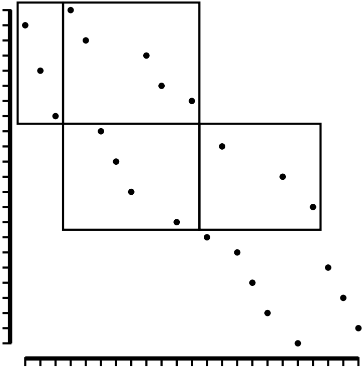

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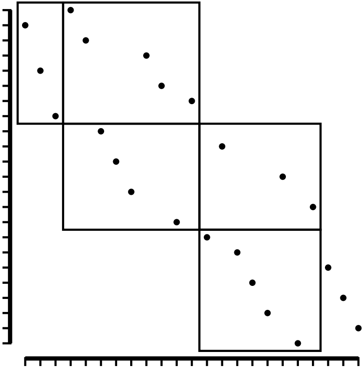

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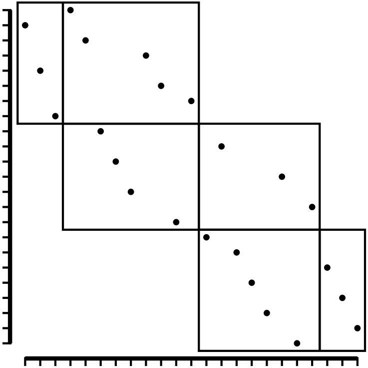

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We find a recurrence for the generating function at each step, where filled-in dots are represented by x and hollow dots are represented by y.

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We have the option of placing a hollow dot above or to the left of the first entry.

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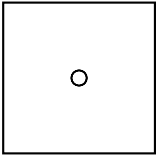

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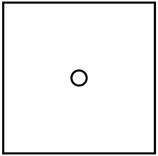

$\Longrightarrow$

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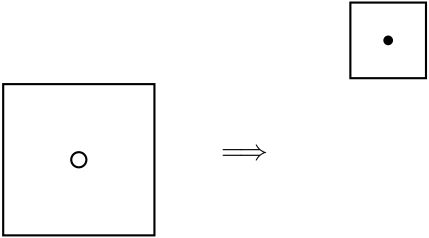

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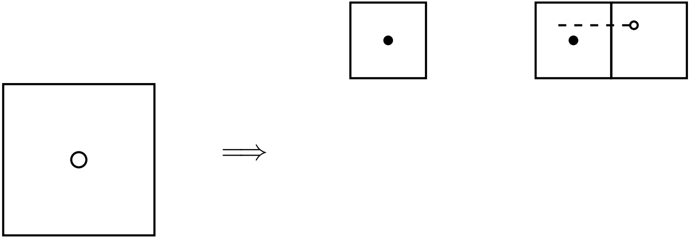

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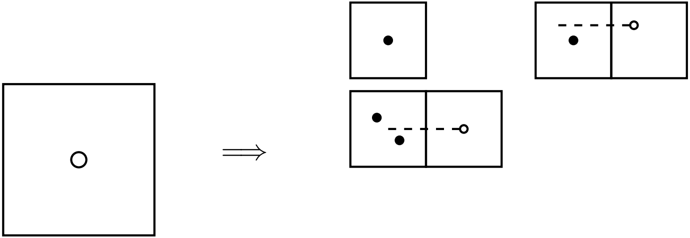

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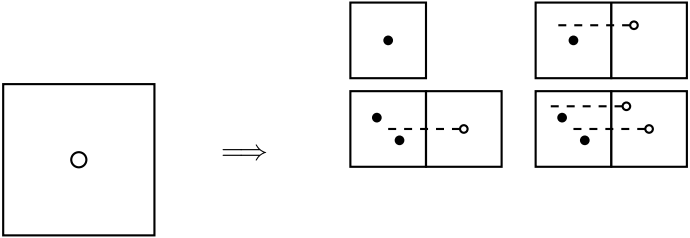

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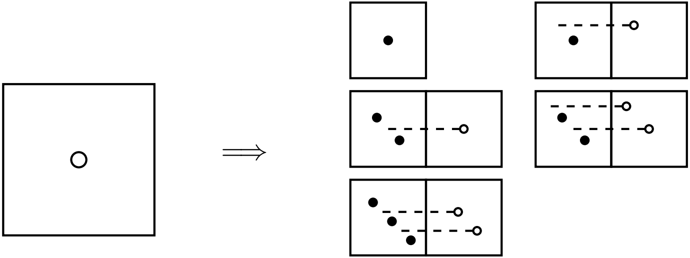

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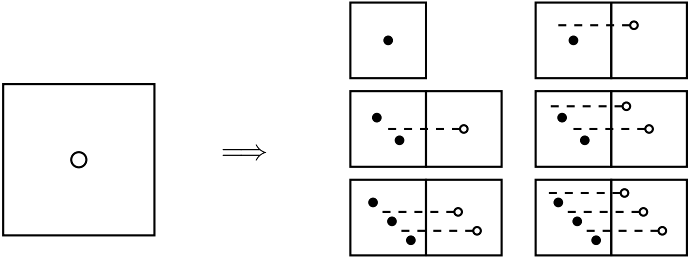

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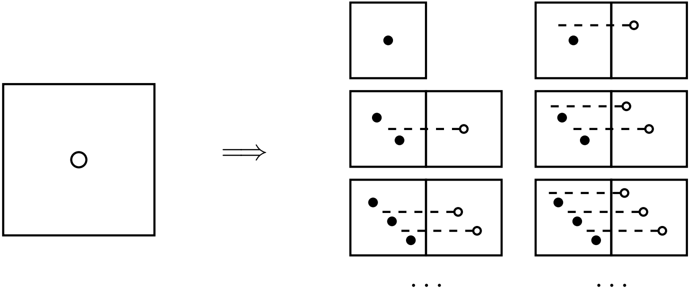

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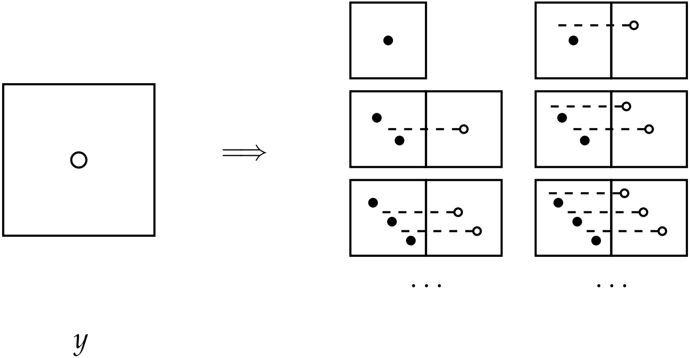

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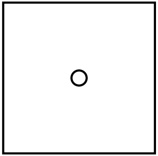

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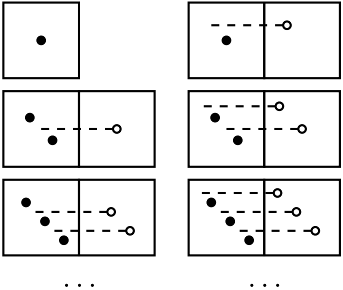
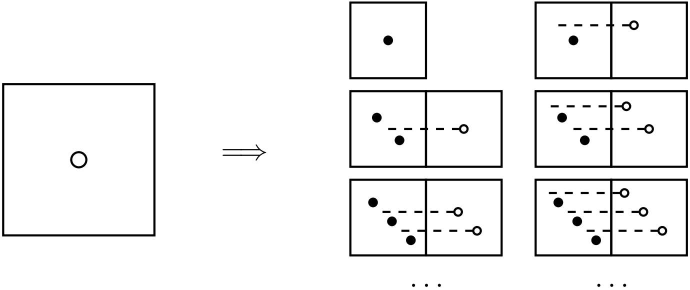


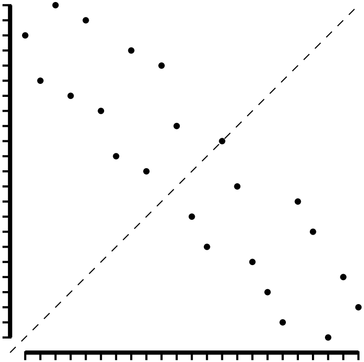
$y \qquad \Longrightarrow$

THE STAIRCASE DECOMPOSITION



$$y \implies x + xy + x^2y + x^2y^2 + \cdots = \frac{x(1+y)}{1-xy}$$

THE STAIRCASE DECOMPOSITION



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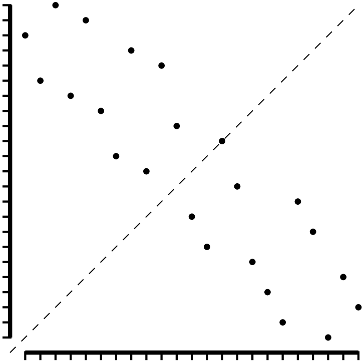

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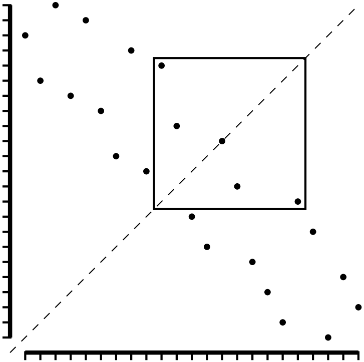

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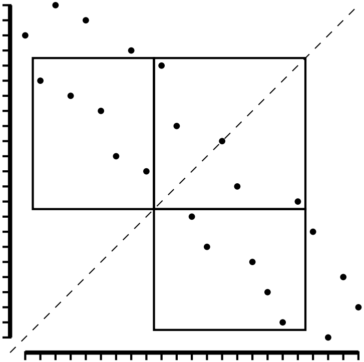

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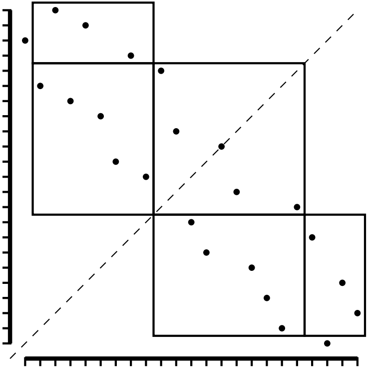

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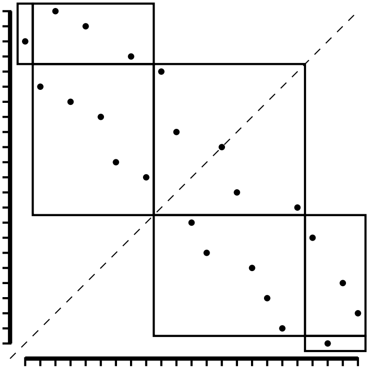

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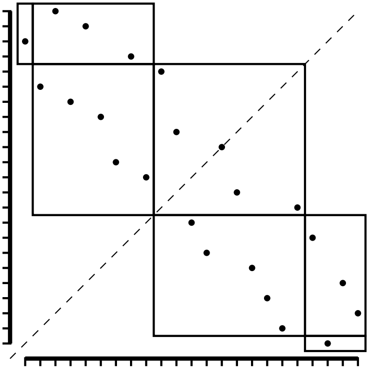

This lets us find: $$\sum_{\substack{\sigma \in \mathrm{Av}^I(123) \\ \sigma \text{ simple}}} u^{\mathrm{lrmin}(\sigma)} v^{\mathrm{rlmax}(\sigma)} z^{\mathrm{fp}(\sigma)}.$$

1342-Avoiding Involutions

Accounting for the involution of length 1, the sum and skew decomposable involutions, and the inflations of simple involutions of length at least 4, we find the generating function

$$f(x) = \frac{x\left(1 - 2x + x^2 + \sqrt{1 - 6x^2 + x^4}\right)}{2\left(1 - 3x + x^2\right)}$$

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which gives the growth rate

$$1 + \frac{1 + \sqrt{5}}{2} \approx 2.62.$$

2341-Avoiding Involutions

Accounting for the involution of length 1, the sum and skew decomposable involutions, and the inflations of simple involutions of length at least 4, we find a generating function $f(x)$ which is algebraic, but very long.

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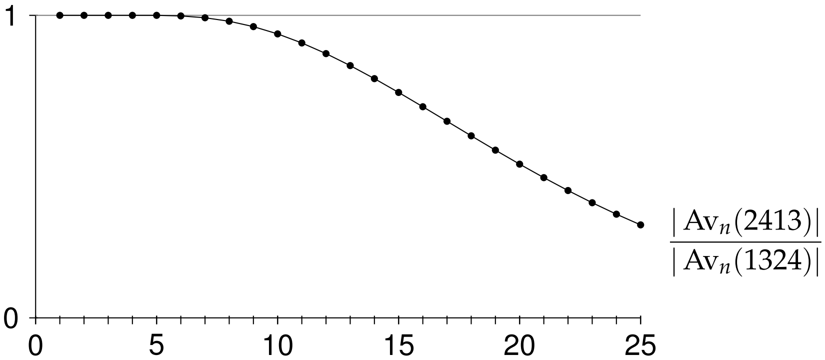
Accounting for the involution of length 1, the sum and skew decomposable involutions, and the inflations of simple involutions of length at least 4, we find a generating function $f(x)$ which is algebraic, but very long.

The growth rate is an algebraic number of degree 16 with minimal polynomial

$$x^{16}-6x^{15}+4x^{14}+50x^{13}-141x^{12}+55x^{11}+326x^{10}-514x^{9}-26x^{8}+725x^{7}$$
$$-561x^{6}-223x^{5}+540x^{4}-206x^{3}-113x^{2}+120x-32$$

and is approximately 2.54.

	2431	**2341**	**1342**	**1234**	**1324**	**3421**	**4231**	**2413**
$\lvert Av^I_5(\beta)\rvert$	24	25	24	21	21	25	21	24
$\lvert Av^I_6(\beta)\rvert$	62	66	62	51	51	66	51	64
$\lvert Av^I_7(\beta)\rvert$	154	170	156	127	126	173	128	166
$\lvert Av^I_8(\beta)\rvert$	396	441	406	323	321	460	327	456
$\lvert Av^I_9(\beta)\rvert$	992	1124	1040	835	820	1218	858	1234
$\lvert Av^I_{10}(\beta)\rvert$	2536	2870	2714	2188	2160	3240	2272	3454
$\lvert Av^I_{11}(\beta)\rvert$	6376	7273	7012	5798	5654	8602	6146	9600
$\lvert Av^I_{12}(\beta)\rvert$	16238	18477	18322	15511	15272	22878	16716	27246
$\lvert Av^I_{13}(\beta)\rvert$	40914	46825	47560	41835	40758	60794	46246	77132
$\lvert Av^I_{14}(\beta)\rvert$	103954	118917	124358	113634	112280	161668	128414	221336
$\lvert Av^I_{15}(\beta)\rvert$	262298	301734	323708	310572	304471	429752	361493	635078
$\lvert Av^I_{16}(\beta)\rvert$	665478	766525	846766	853467	852164	1142758	1020506	1839000
$\lvert Av^I_{17}(\beta)\rvert$	1680726	1946293	2208032	2356779	2341980	3038173	2913060	5331274
$\lvert Av^I_{18}(\beta)\rvert$	4260262	4944614	5777330	6536382	6640755	8078606	8335405	15555586
$\lvert Av^I_{19}(\beta)\rvert$	10766470	12557685	15082372	18199284	18460066	21479469	24067930	45465412
$\lvert Av^I_{20}(\beta)\rvert$	27274444	31900554	39469786	50852019	52915999	57113888	69646035	133517130
growth rate	?	≈ 2.54	≈ 2.62	3	$(3.13, 4.84)$	?	?	≈ 3.15


1
0
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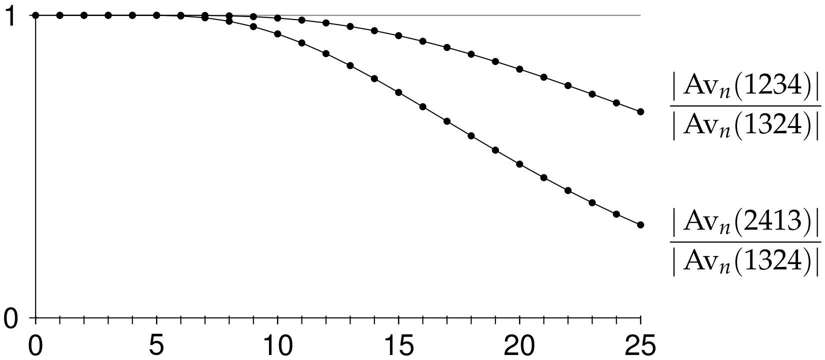
15
20
25
$\frac{|\operatorname{Av}_n(2413)|}{|\operatorname{Av}_n(1324)|}$


1
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$\frac{|\text{Av}_n(1234)|}{|\text{Av}_n(1324)|}$
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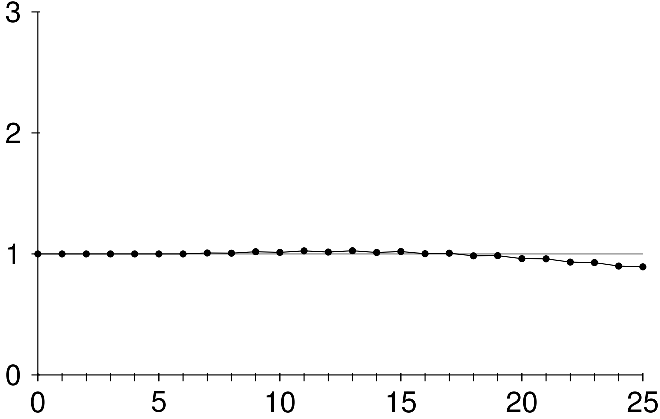
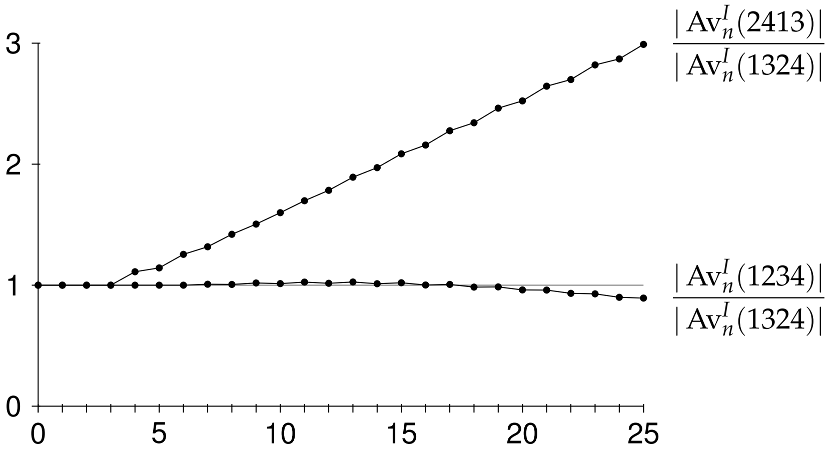


$$\frac{|\,\mathrm{Av}_n^I(1234)|}{|\,\mathrm{Av}_n^I(1324)|}$$


3
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10
15
20
25
$\frac{|\text{Av}_n^I(2413)|}{|\text{Av}_n^I(1324)|}$
$\frac{|\text{Av}_n^I(1234)|}{|\text{Av}_n^I(1324)|}$

Conjectures

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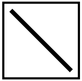

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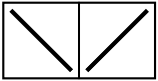

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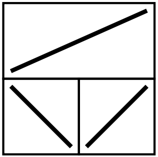

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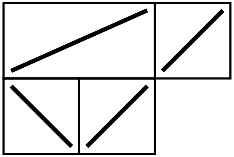
It seems like the simple permutations in $Av^I(2431)$ may be those that can be drawn on the diagram



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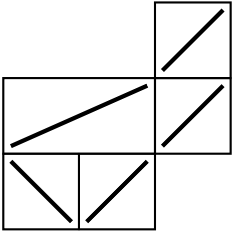

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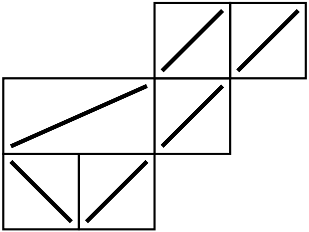

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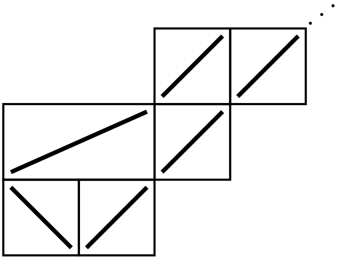

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By calculation, it seems that

$$\text{Simples}\left(\mathrm{Av}^I(2431)\right) = \text{Simples}\left(\mathrm{Av}^I(1432, 2431, 4132, 4231)\right).$$

There is a close connection between these numbers and the Motzkin numbers.

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Since 4321 is in the basis, these permutations can be decomposed as the union of three increasing permutations.

Conjectures

The set $\mathrm{Av}^I(3421)$ appears to have the generating function

$$\frac{1 - x - 4x^3 + (1-x)\sqrt{1-4x^2}}{2(1-x)(1-2x-x^2-2x^3)}$$

(agrees up to 25 terms), which yields a growth rate of ≈ 2.66 and so is overtaken by $\mathrm{Av}^I(1234)$ at some point.

	2431	2341	1342	3421	1234	1324	4231	2413
$\|\mathrm{Av}^I_{13}(\beta)\|$	40914	46825	47560	60794	41835	40758	46246	77132
$\|\mathrm{Av}^I_{14}(\beta)\|$	103954	118917	124358	161668	113634	112280	128414	221336
$\|\mathrm{Av}^I_{15}(\beta)\|$	262298	301734	323708	429752	310572	304471	361493	635078
$\|\mathrm{Av}^I_{16}(\beta)\|$	665478	766525	846766	1142758	853467	852164	1020506	1839000
$\|\mathrm{Av}^I_{17}(\beta)\|$	1680726	1946293	2208032	3038173	2356779	2341980	2913060	5331274
$\|\mathrm{Av}^I_{18}(\beta)\|$	4260262	4944614	5777330	8078606	6536382	6640755	8335405	15555586
$\|\mathrm{Av}^I_{19}(\beta)\|$	10766470	12557685	15082372	21479469	18199284	18460066	24067930	45465412
$\|\mathrm{Av}^I_{20}(\beta)\|$	27274444	31900554	39469786	57113888	50852019	52915999	69646035	133517130
$\|\mathrm{Av}^I_{21}(\beta)\|$	68956648	81021172	103120888	151859593	142547559	148551532	203046400	392841336
$\|\mathrm{Av}^I_{22}(\beta)\|$	174619096	205805457	269892878	403792010	400763223	429756305	593174577	1160033656
$\|\mathrm{Av}^I_{23}(\beta)\|$	441605616	522718064	705520028	1073654591	1129760415	1216856079	1743666189	3432015726
$\|\mathrm{Av}^I_{24}(\beta)\|$	1117997364	1327735500	1846667206	2854820376	3192727797	3548148478	5135622924	10182891552
$\|\mathrm{Av}^I_{25}(\beta)\|$	2827910868	3372334557	4829188144	7590820575	9043402501	10122789960		30267591290

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Thanks for coming. Any questions?