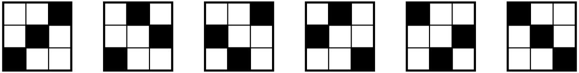
Sorting with $\mathcal{C}$-Machines

(joint work with Michael Albert, Cheyne Homberger, Nathaniel Shar, and Vince Vatter)

Jay Pantone
Dartmouth College
Hanover, NH



University of Melbourne, Victoria

July 27, 2015

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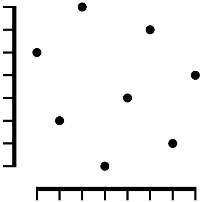

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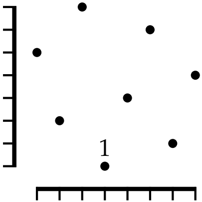

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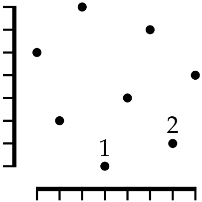

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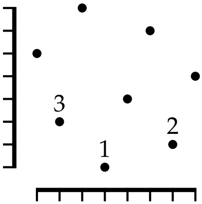

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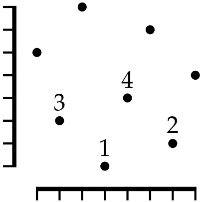

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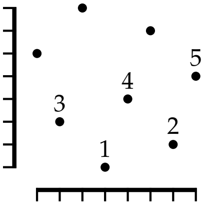

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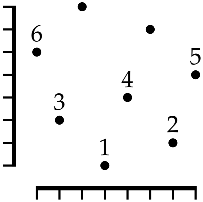

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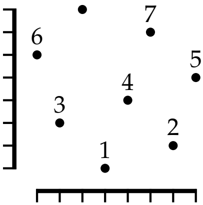

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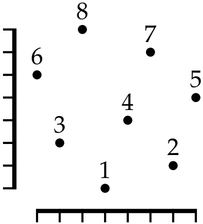

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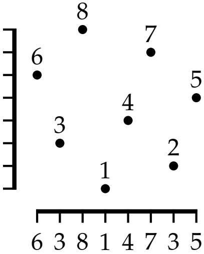

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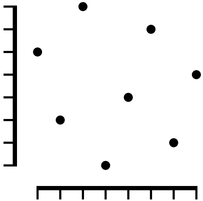

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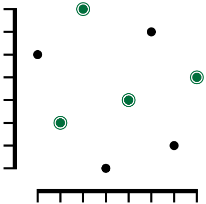

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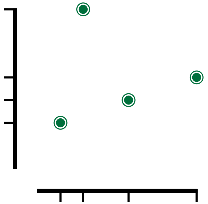

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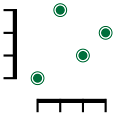

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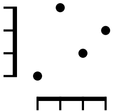

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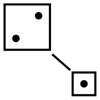

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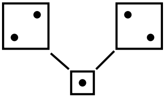

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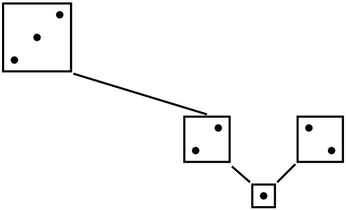

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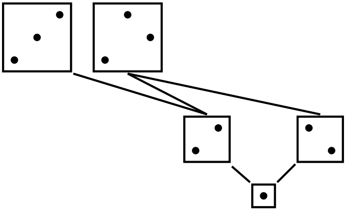

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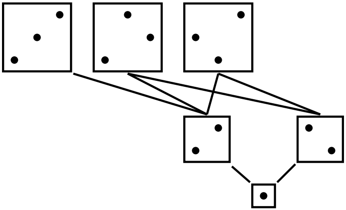

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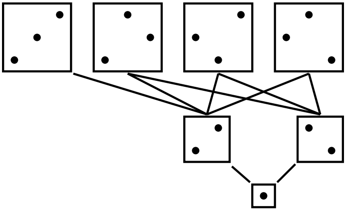

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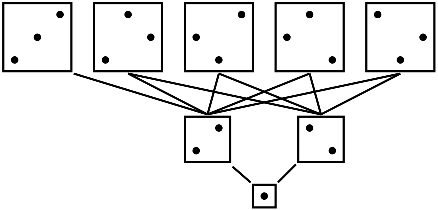

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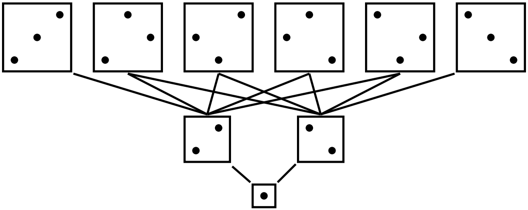

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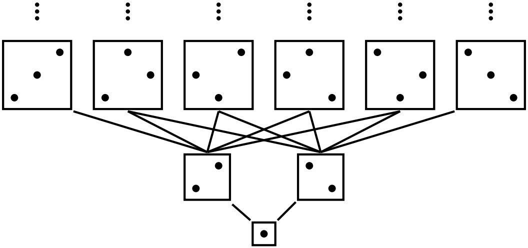

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The class with basis B is denoted $\mathrm{Av}(B)$.

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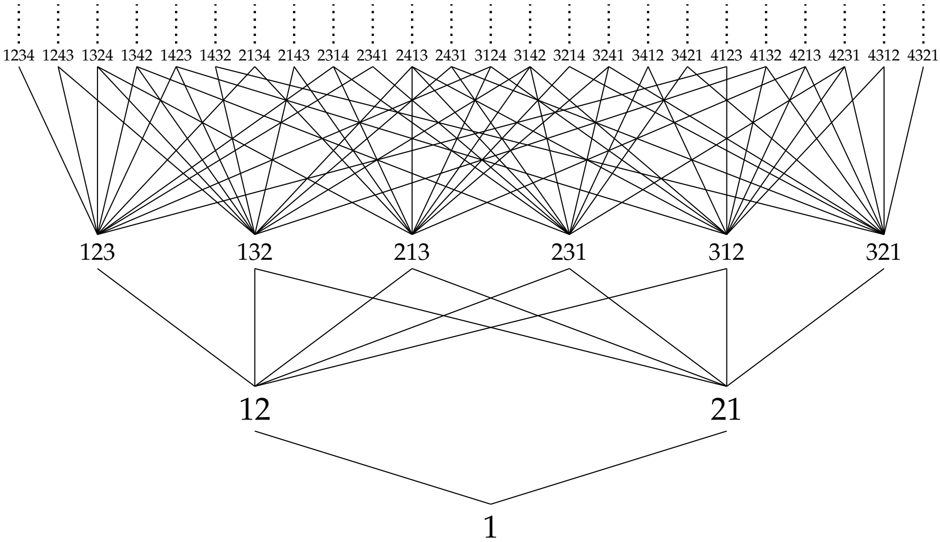

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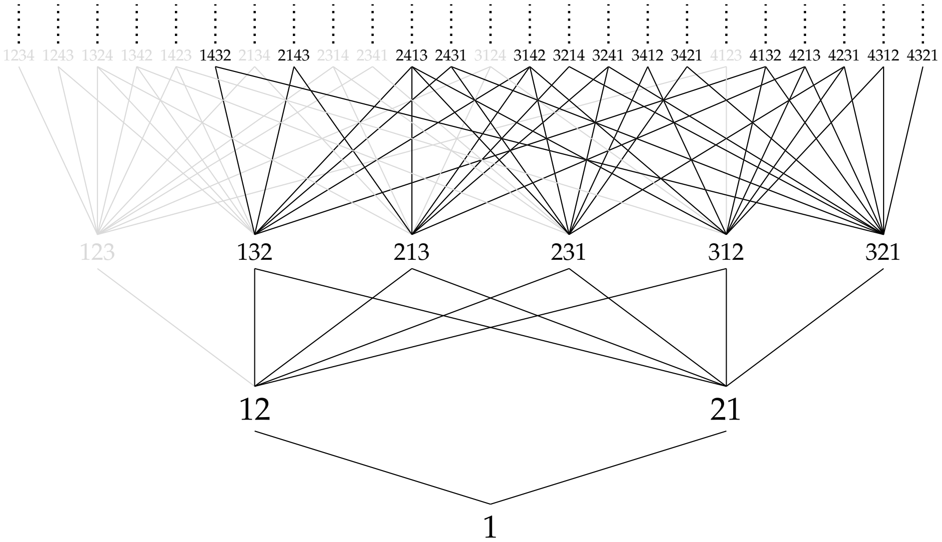

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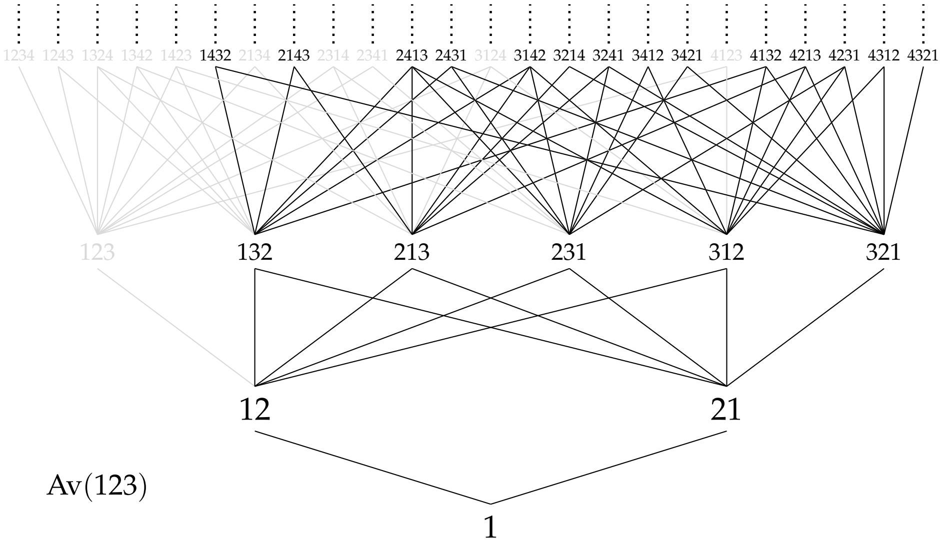

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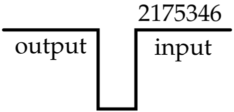

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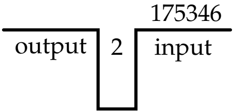

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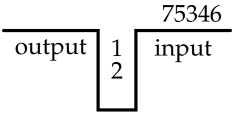

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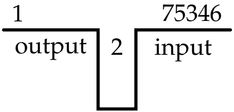

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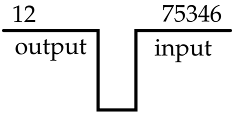

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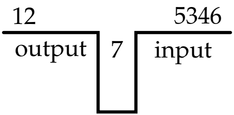

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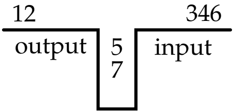

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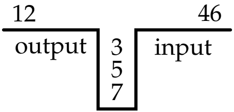

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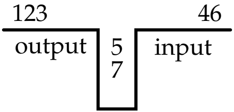

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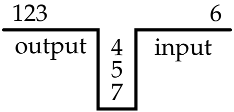

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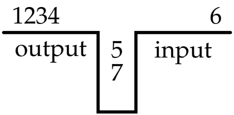

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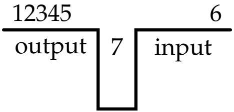

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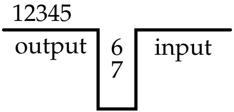

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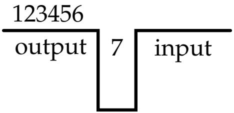

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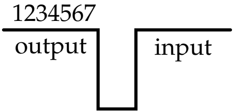

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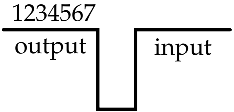

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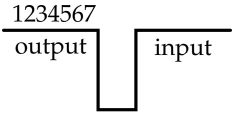

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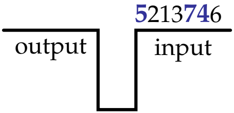

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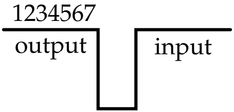

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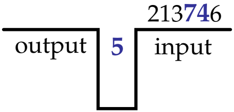

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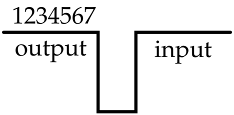

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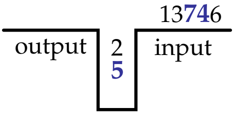

Sorting with a Stack

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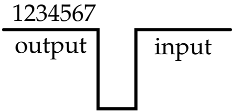

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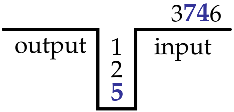

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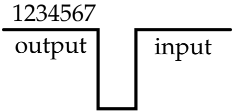

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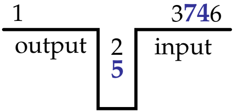

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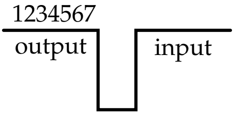

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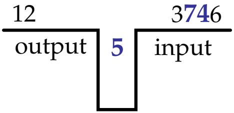

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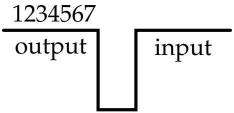

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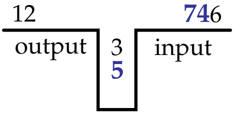

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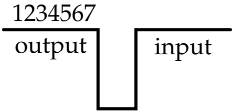

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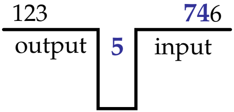

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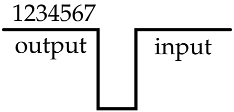

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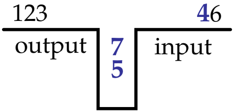

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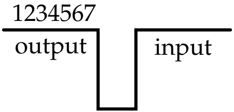

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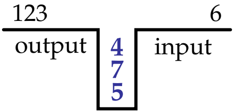

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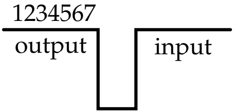

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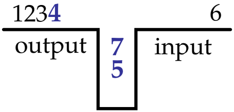

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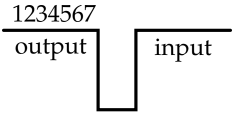

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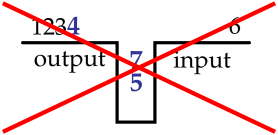

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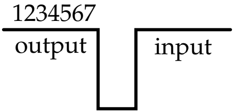

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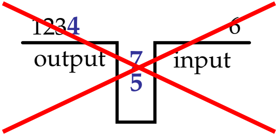
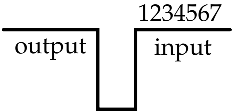


Permutations containing 231 cannot be sorted by a stack. In fact, the permutations that are stack-sortable are exactly those in Av(231).

GENERATING WITH A STACK

The permutation 2156473 can be generated by a stack.



(Notice the stack always holds a decreasing subpermutation when read top to bottom.)

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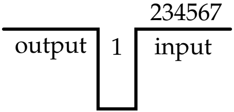

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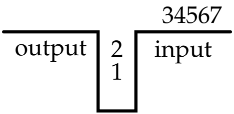

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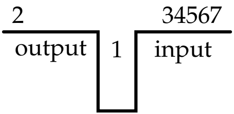

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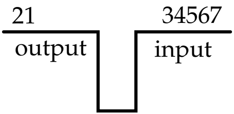

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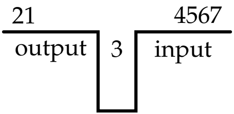

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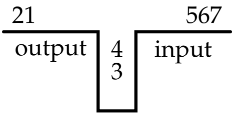

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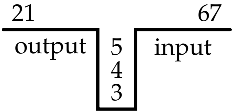

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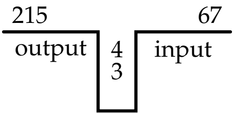

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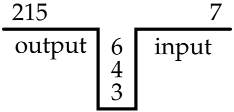

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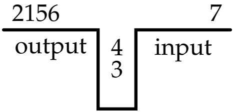

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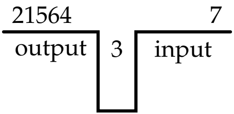

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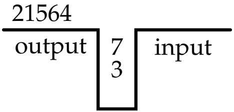

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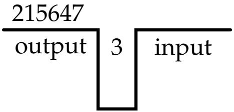

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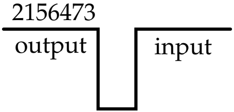

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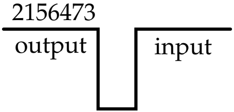

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The permutations that can be *generated* by a stack are exactly the inverses of those that can be *sorted* by a stack.

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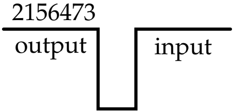

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A stack can generate the permutations in $\mathrm{Av}(231^{-1}) = \mathrm{Av}(312)$.

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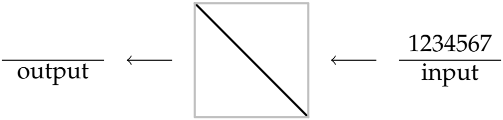

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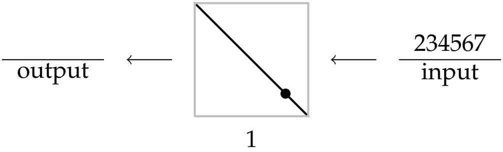

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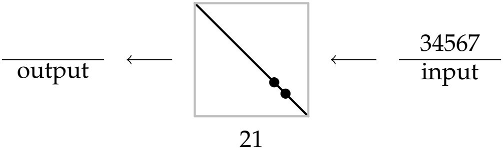

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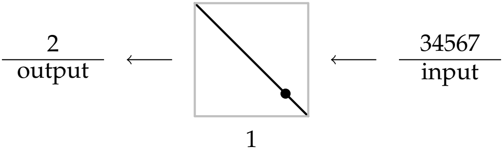

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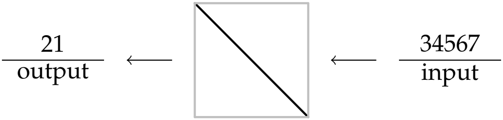

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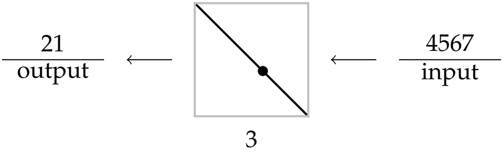

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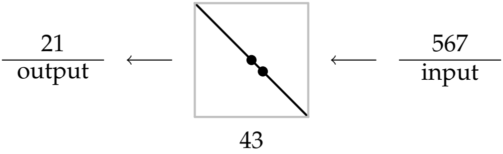

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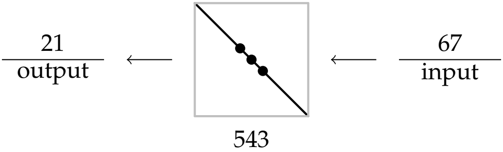

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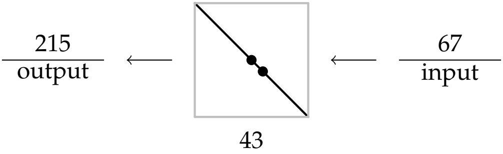

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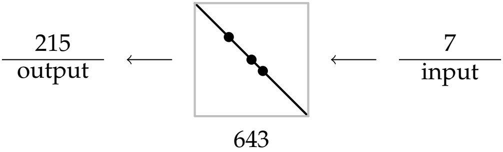

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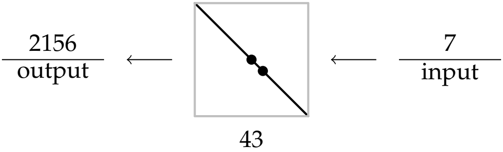

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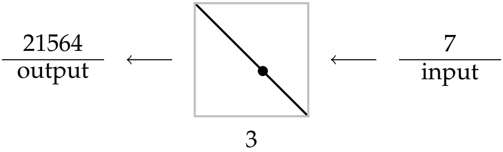

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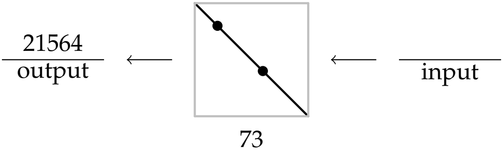

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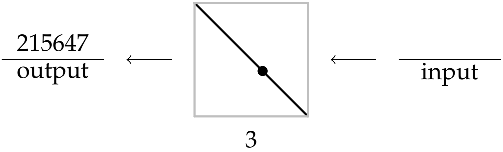

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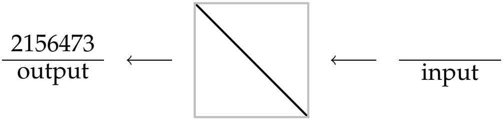

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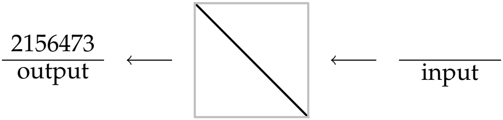

Since the container holds permutations that avoid the pattern 12, we call this the Av(12)-machine.

Generalizing Stacks

We allow a third operation: an entry can bypass the container and move straight from the input to the output.

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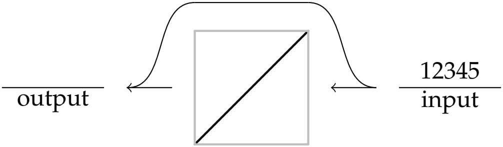

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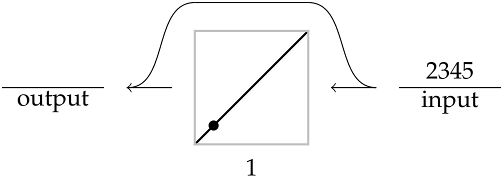

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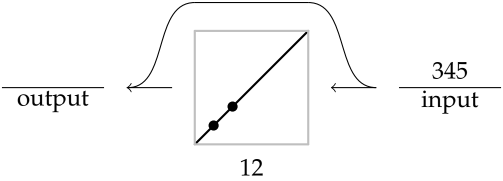

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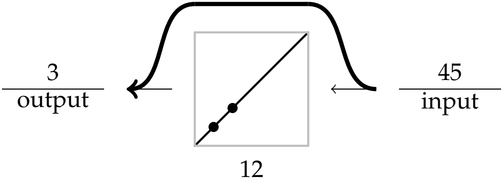

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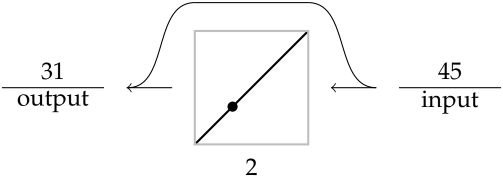

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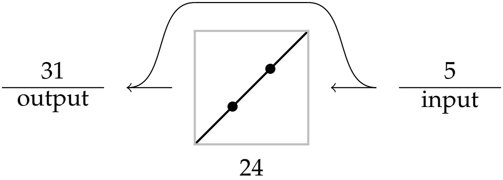

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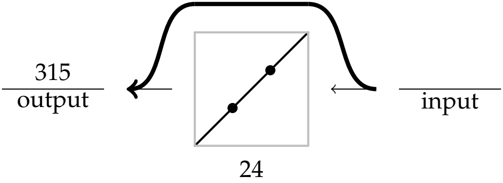

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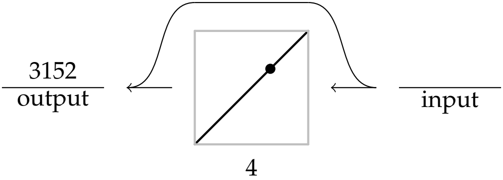

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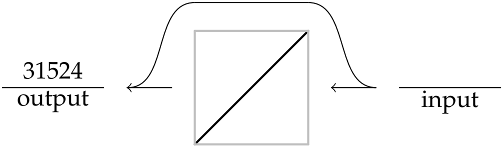

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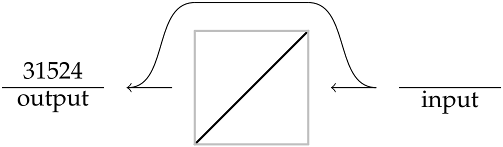

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Theorem. The Av(B)-machine generates the class

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where ${}^{+}\beta$ is formed by adding a new maximum in front of β.

Example: The Av(123, 4132)-machine generates the class Av(4123, 54132).

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The operation sequence of the Av(12)-machine makes it easy to find the generating function for Av(312).

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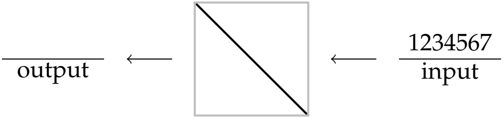

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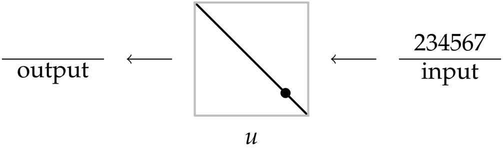

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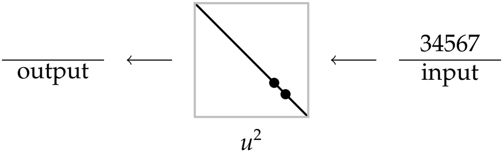

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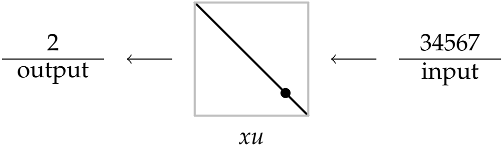

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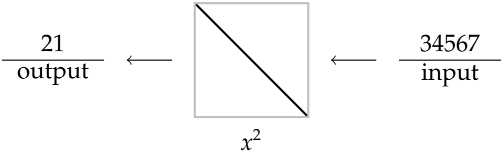

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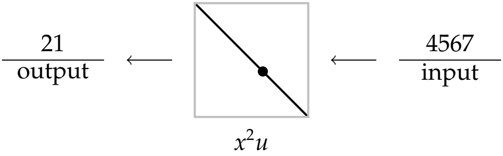

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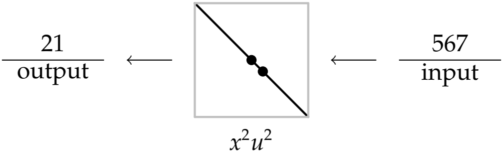

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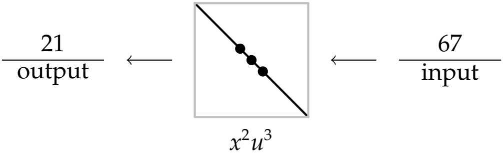

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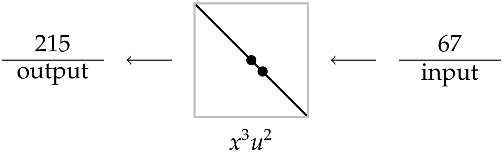

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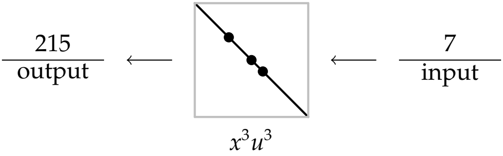

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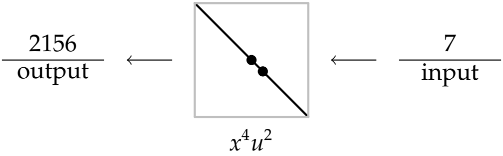

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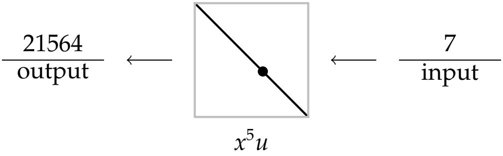

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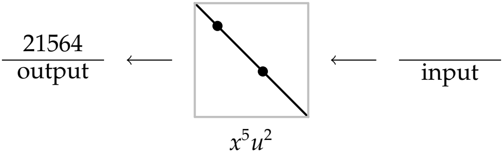

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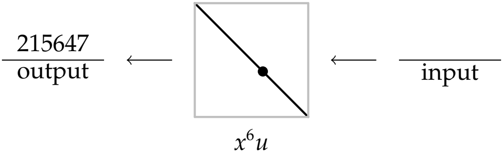

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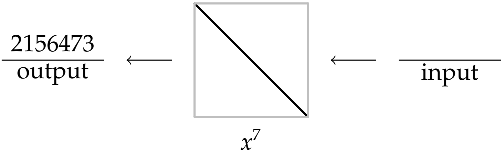

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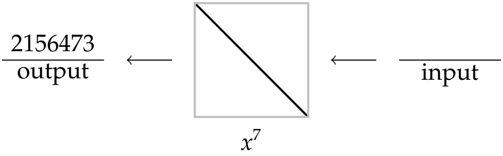

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Hence the generating function for these states is $f(x,0)$.

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$$f(x,u) = 1 + uf(x,u) + \frac{x}{u}\left(f(x,u) - f(x,0)\right)$$

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Solving,

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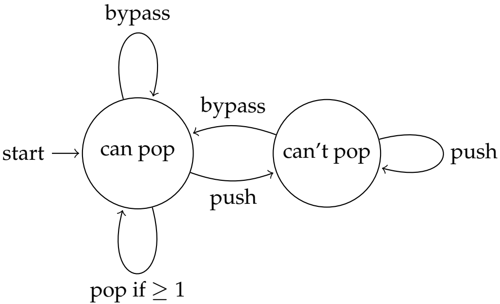

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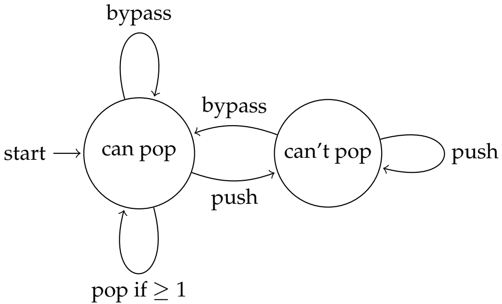

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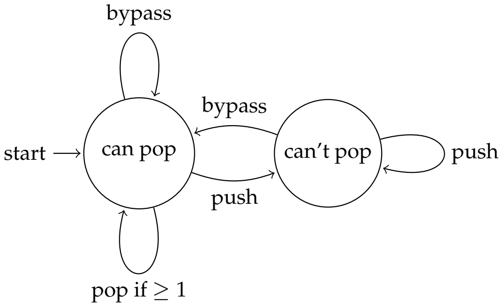

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Solve $g(x,u)$ in terms of $f(x,u)$ → substitute into $f(x,u)$ → kernel method → same answer!

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Some of these 56 $2{\times}4$ classes have the same enumeration despite being non-isomorphic. There are precisely 38 different enumerations for the $2{\times}4$ classes.

ENUMERATING THE SCHRÖDER CLASSES

A *2×4 class* is a permutation class with two basis elements of length 4.

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Of these 38 classes, the exact enumerations are known for 29 of them. (More on this later!)

ENUMERATING THE SCHRÖDER CLASSES

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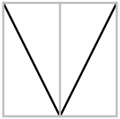
Every permutation in $\mathrm{Av}(231, 132)$ is a descending segment followed by an ascending segment, and so we say that the class has the shape



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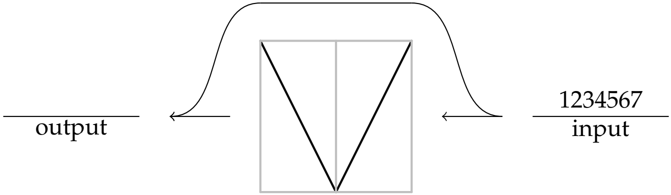
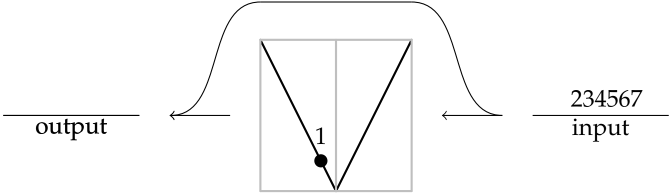


Whenever the machine is nonempty, there are always two places to push a new entry.

ENUMERATING THE SCHRÖDER CLASSES



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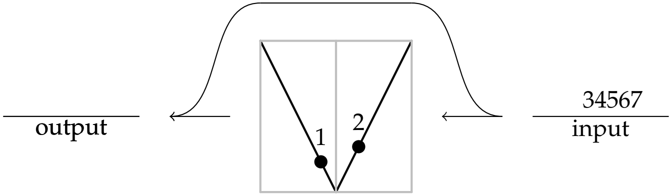

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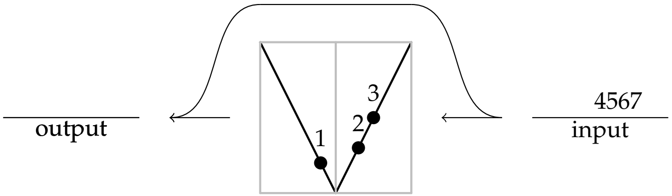

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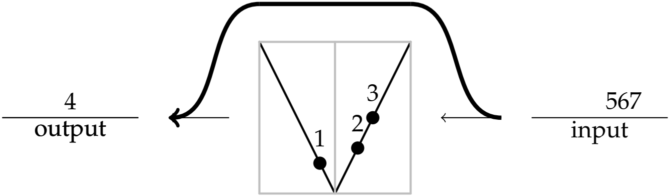

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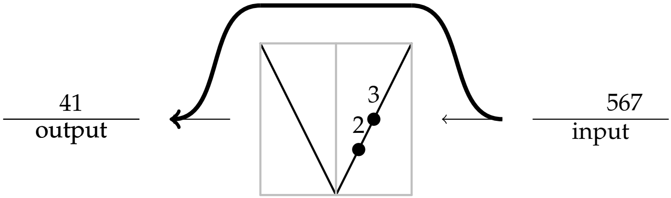

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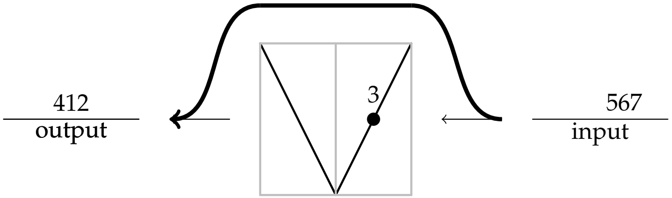

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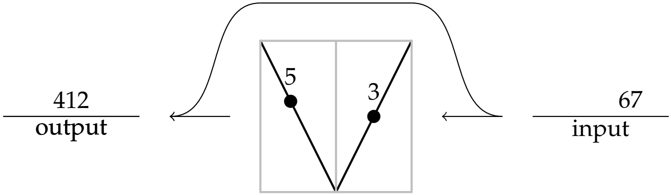

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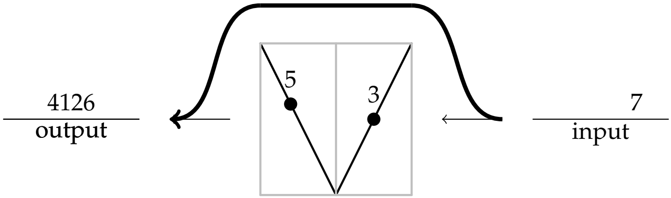

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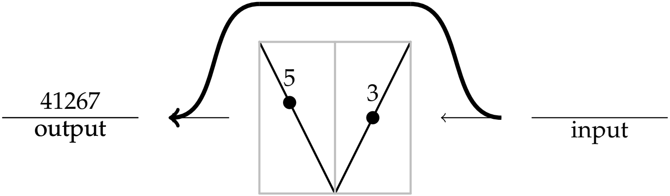

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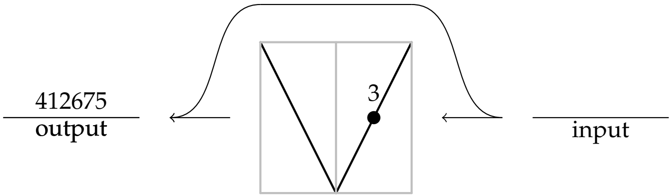

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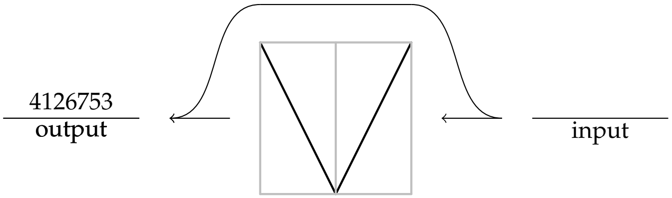

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ENUMERATING THE SCHRÖDER CLASSES

A small adjustment to the earlier functional equations gives:

$$f(x,u) = 1 + x(f(x,u) + g(x,u)) + \frac{x}{u}(f(x,u) - f(x,0)),$$
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An application of the kernel method gives the generating function for the Schröder numbers.

$$f(x,0) = \frac{3 - x - \sqrt{1 - 6x + x^2}}{2}$$

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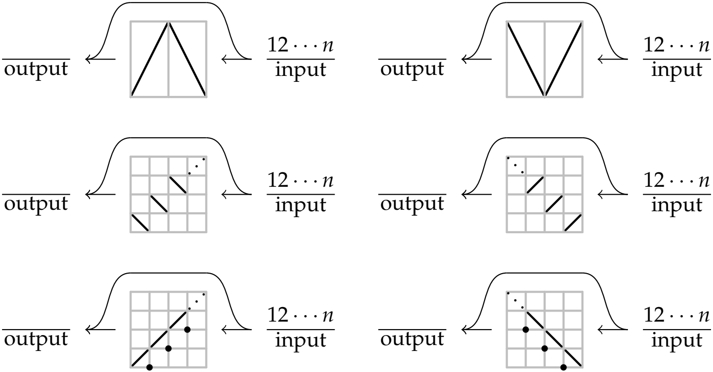

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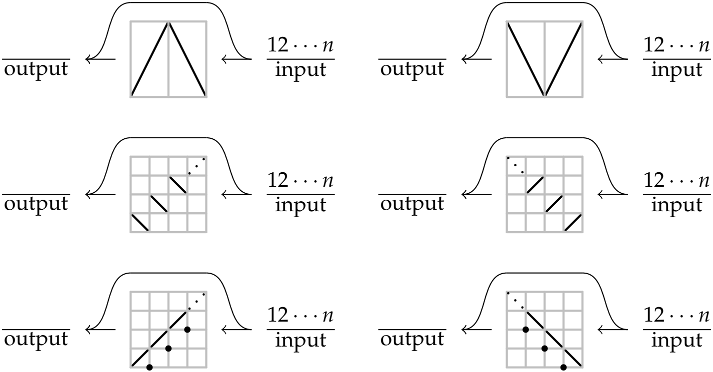

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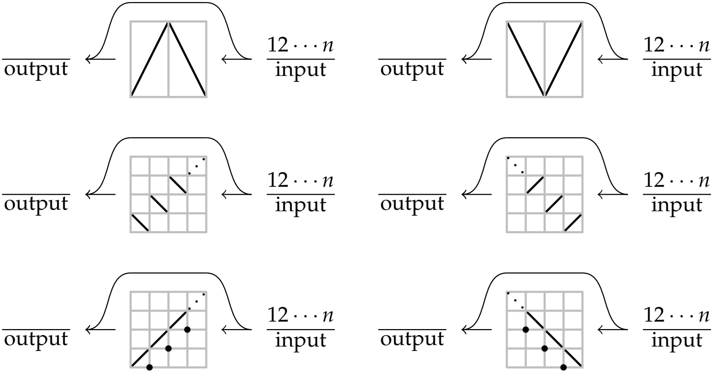

All are enumerated by the Schröder numbers.

Even better, the push-pop-bypasses sequences automatically give a bijection between any pair of these classes.

THE FIBONACCI MACHINES

There are exactly two permutation classes whose enumeration is given by the Fibonacci numbers.

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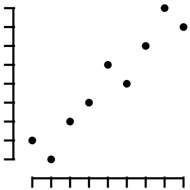
- $\mathcal{F}_{\oplus} = \mathrm{Av}(231, 312, 321)$: 1 and 21 patterns stacked $\nearrow$



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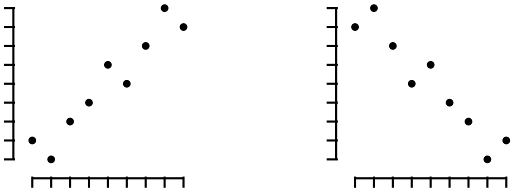
- $\mathcal{F}_\oplus = \text{Av}(231, 312, 321)$: 1 and 21 patterns stacked $\nearrow$
- $\mathcal{F}_\ominus = \text{Av}(123, 132, 213)$: 1 and 12 patterns stacked $\nwarrow$



THE FIBONACCI MACHINES

By the basis theorem, the $\mathcal{F}_\oplus$-machine generates the class

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These two enumerations are new, and they demonstrate the power of the $\mathcal{C}$-machine model.

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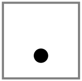

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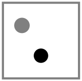

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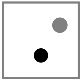

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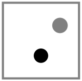

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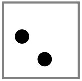

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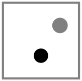

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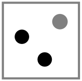

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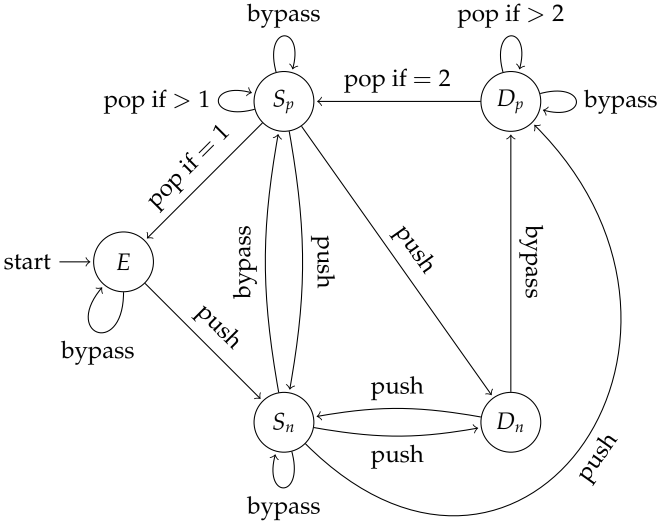

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The automaton turns into functional equations:

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- Guess their generating functions.
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- Verify the guesses.
- The solution is now rigorous.

ENUMERATING THE $\mathcal{F}_{\oplus}$-MACHINE

We find that the class generated by the $\mathcal{F}_{\oplus}$-machine has an algebraic generating function $E(x)$ that satisfies the minimal polynomial:

$$\begin{aligned}(2x^2+8x-1)E(x)^4+(x^3+4x^2-46x+5)E(x)^3\\ +(3x^3-21x^2+94x-9)E(x)^2\\ +(x^3+12x^2-82x+7)E(x)\\ +3x^2+26x-2=0\end{aligned}$$

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\end{aligned}
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and exponential growth rate

$$\approx 5.162$$

Enumerating the $\mathcal{F}_{\ominus}$-Machine

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ENUMERATING THE $\mathcal{F}_{\ominus}$-MACHINE

The same approach fails for the $\mathcal{F}_{\ominus}$-machine because now we're pushing and popping on the same layer.

Instead we construct a context-free grammar to represent the allowed push-pop-bypass sequences.

ENUMERATING THE $\mathcal{F}_{\ominus}$-MACHINE

$$
\begin{array}{lcccccccc}
S & \longrightarrow & \epsilon & | & xS & | & (+w)W_n(-w)S & & \\
W_p & \longrightarrow & \epsilon & | & xW_p & | & (+w)W_n(-w)W_p & | & (+r)R_n(-r)W_p \\
W_n & \longrightarrow & & & xW_p & | & (+w)W_n(-w)W_p & | & (+r)R_n(-r)W_p \\
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This is easily solved to find that the algebraic generating function $s(x)$ for Av(4123, 4132, 4213) has minimal polynomial

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$$1 + (x-1)s(x) - xs(x)^2 + xs(x)^3,$$

and growth rate

$$\frac{67240 + (779\sqrt{57} - 1927)r^{1/3} - (19\sqrt{57} - 457)r^{2/3}}{40344} \approx 5.219,$$

where $r = 1502 + 342\sqrt{57}$.

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For each of these machines, we have been able to set up functional equations (that we can't solve), as well as generate many terms in the counting sequence.

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We have found four simple machines that generate classes whose enumerations are unknown.

For each of these machines, we have been able to set up functional equations (that we can't solve), as well as generate many terms in the counting sequence.

We have tried to guess algebraic, D-finite, and D-algebraic generating functions based on these terms, but have found none.

THE Av(123, 231)-MACHINE

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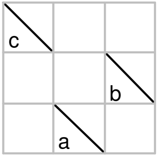

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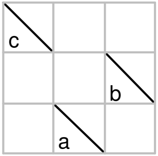

The states of the machine can be represented by 4-tuples (a, b, c, P) where a, b, and c are the number of a, b, c entries, and P is a boolean representing whether we can or can't pop.

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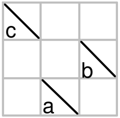

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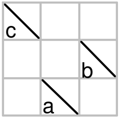

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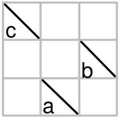

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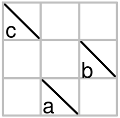

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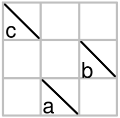

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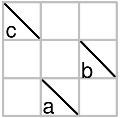

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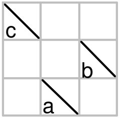
The Av(123, 231)-Machine



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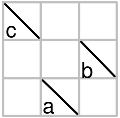

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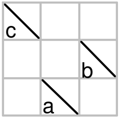

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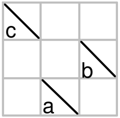

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THE Av(123, 231)-MACHINE

With a little work, we can translate the state transitions into a set of functional equations:

$$\begin{aligned} A(a,x) &= 1 + \frac{x}{a}(A(a,x) - A(0,x)) + aA(a,x) + xB(0,a,x), \\ B(a,b,x) &= \frac{1}{a}(A(a,x) - A(0,x))\frac{bC}{1-b} + B(a,b,x)\frac{bC}{1-b} \\ &\qquad + \frac{x}{a}(1+C)(B(a,b,x) - B(0,b,x)). \end{aligned}$$

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The generating function for the class is $A(0,x)$, but we have no idea how to solve these equations.

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Using the state transitions, dynamic programming, and Amazon cloud computing, we have computed the first 1000 terms of Av(4123, 4231).

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Despite all of this information, we have been unable to determine the nature of the generating function.

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For example, the exponential generating function $B(x)$ for the Bell numbers is D-algebraic because

$$B(x)B'(x) - B(x)B''(x) + B'(x)^2 = 0.$$

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                                                 / 2       \
               /d       \                        |d         |    /d       \2
        F(x) |-- F(x)| - F(x) |--- F(x)| + |-- F(x)|
               \dx      /                        |  2       |    \dx      /
                                                 \dx       /
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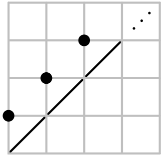
              \dx       /                     |  2      |   \dx      /
                                              \dx       /
```

Despite having 1000 initial terms of Av(4123, 4231), we can't guess any generating function!

THREE MORE EXOTIC MACHINES



600 terms
Av(4231, 4321)

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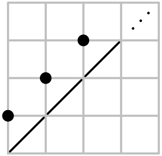

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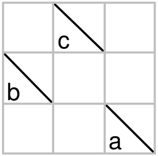

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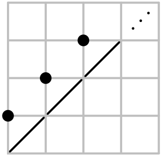

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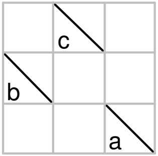

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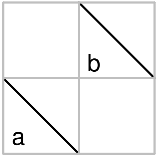

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UNSOLVED MYSTERIES

Conway and Guttmann analyzed 36 terms of the counting sequence of Av(4231), which is also suspected to have a non-D-finite generating function and found that the sequence exhibited strange behavior.

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We are in the process of exploring the asymptotic behavior of these four exotic classes in a similar way.

Thanks for coming! Any questions?