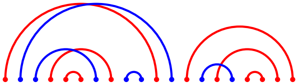
Patterns and Colorability in Chord Diagrams

Jay Pantone

Dartmouth College

Hanover, NH



Applied and Computational Combinatorics

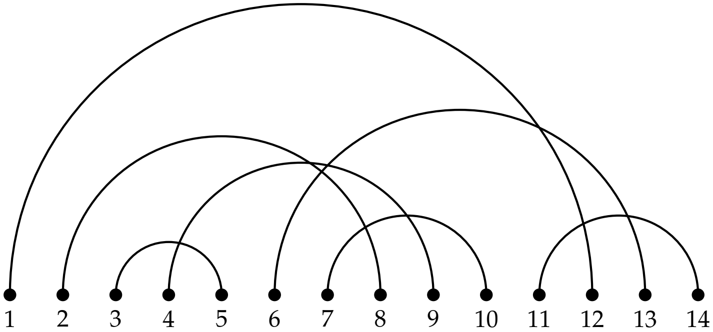
JMM 2018 San Diego, CA

January 10, 2018

joint work with Peter Doyle and Everett Sullivan

Chord Diagrams

A *chord diagram* with n chords is a pairing of $2n$ points in a line.



Chord Diagrams

Other names: *matchings*, *arc systems*, . . .

Chord Diagrams

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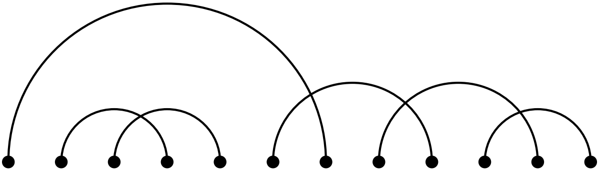
Applications to **sorting permutations with stacks**,
RNA folding,
knot theory, ...

Two Types of Patterns

- 2-colorability



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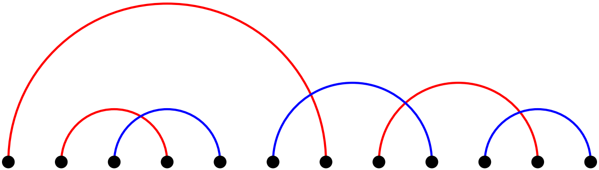

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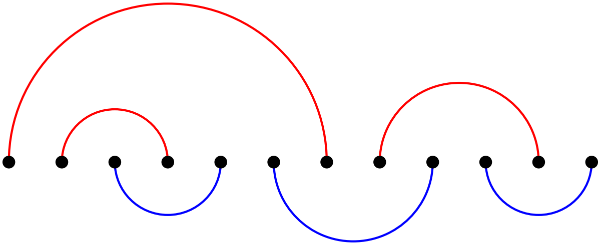

Two Types of Patterns

- min-length-k: all chords have length $\geq k$

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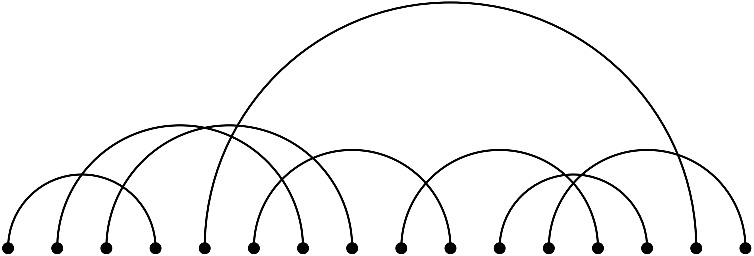
min-length-3 of size 4:



Enumeration Recipes

2-Colorable

Ingredients:

- ≫ structural theorem
- ≫ functional equations
- ≫ symbolic manipulation
- ≫ differential approximation

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Min-Length-k

Ingredients:
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- finite state machines
- transfer matrix method
- sieve method
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- creative telescoping
- guess-and-check

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~~Min-Length-*k*~~

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Ingredient 1: Structural Theorem

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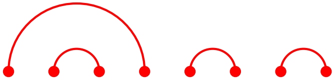

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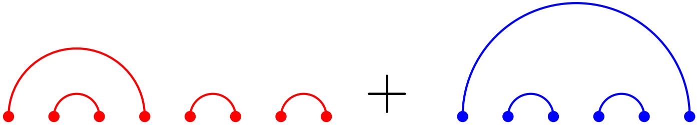

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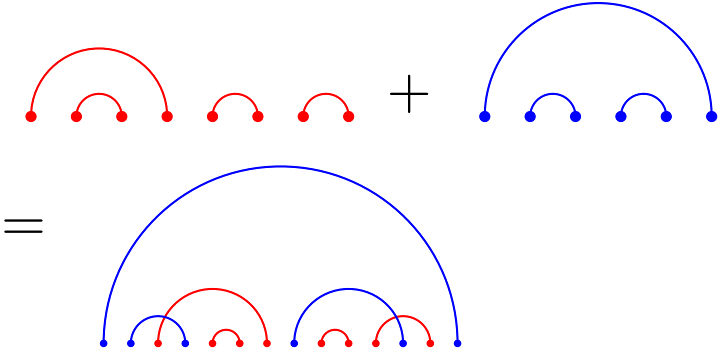

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$B(z)$ satisfies the linear differential equation

$$0 = 12z + (12z - 2)B(z) + (48z^2 - 4z)B'(z) + (16z^3 - z^2)B''(z).$$

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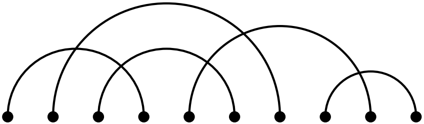

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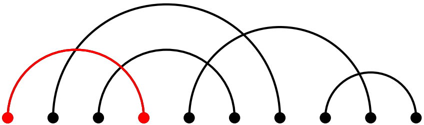

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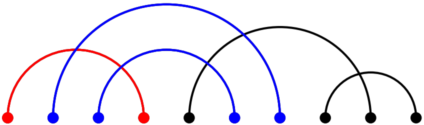

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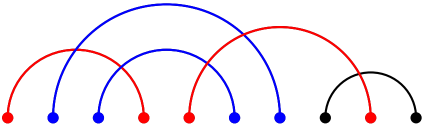

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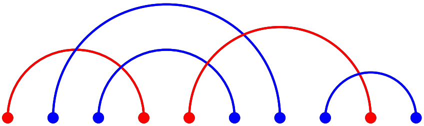

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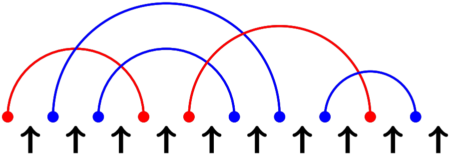

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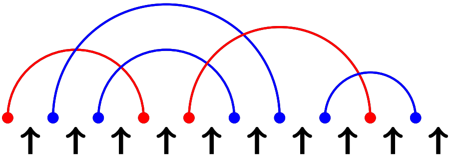

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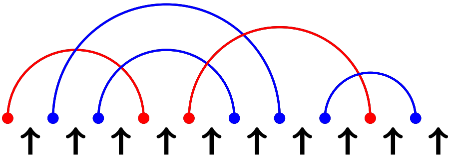

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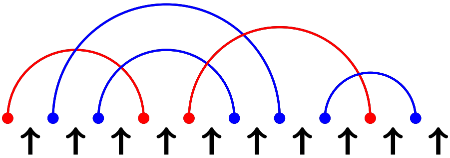

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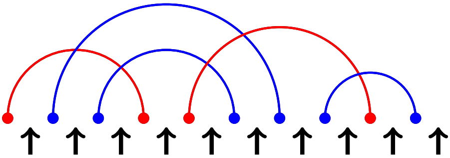

$$B(z) = 2\,C(\;z\,(1 + B(z))^2\;)$$

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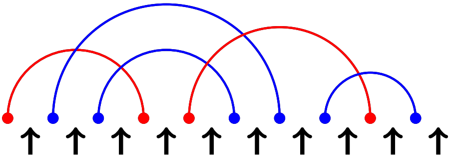

$$B(z) = 2\,C(\ \ z\,(1 + B(z))^2\ \ )$$

This implicitly defines $C(z)$ in terms of $B(z)$.

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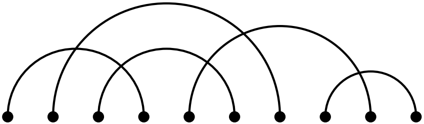

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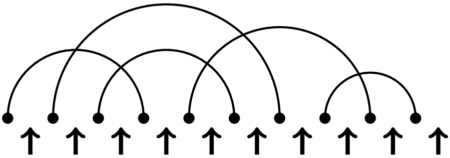

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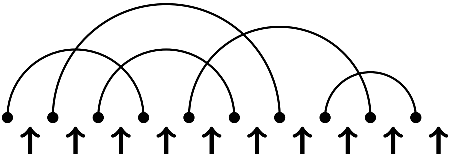

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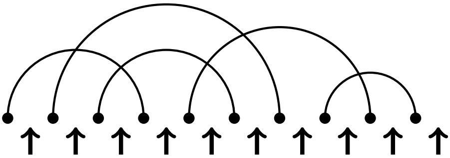

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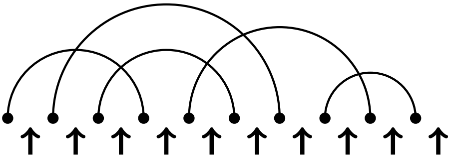

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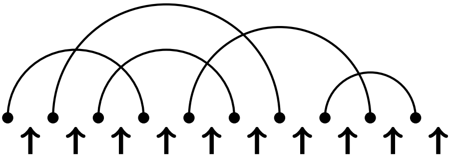

$$F(z) = C(\ \ z\,(1 + F(z))^2\ \ )$$

This implicitly defines $F(z)$ in terms of $C(z)$.

Ingredient 2: Functional Equations

- Define $B(z)$ implicitly by

$$0 = 12z + (12z - 2)B(z) + (48z^2 - 4z)B'(z) + (16z^3 - z^2)B''(z).$$

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Then $F(z) = z + 3z^2 + 14z^3 + 84z^4 + \cdots$ is the generating function for 2-colorable chord diagrams.

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$$\begin{aligned}0 = &(-8+12z)\,F(z)^7 + (-12z+20)\,F(z)^6 + (3z-18)\,F(z)^5 - F(z)^3 - 8z^3F'(z)^3 + 7F(z)^4\\ &- 12z^2F(z)\,F'(z)^2 + 6z\,(3z-10)\,F(z)^4F'(z) + z^2F(z)^3F''(z)/2 + 6z^3\,(4z-3)\,F(z)^2F'(z)^3\\ &- 8z^2\,(8z-3)\,F(z)^4F'(z)^2 + 32zF(z)^3F'(z) + 4z^2\,(4z-1)\,F(z)^6F''(z)\\ &- 12z\,(5z-4)\,F(z)^5F'(z) - 2z^2\,(4z-3)\,F(z)^5F''(z) + 16z\,(3z-1)\,F(z)^6F'(z)\\ &- 6zF(z)^2F'(z) + 42z^2F(z)^2F'(z)^2 + 12z^2\,(3z-4)\,F(z)^3F'(z)^2 - 8z^3\,(4z-1)\,F(z)^3F'(z)^3\\ &+ 15z^3F(z)\,F'(z)^3 - 3z^2F(z)^4F''(z).\end{aligned}$$

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$$- 12z^2F(z)\,F'(z)^2 + 6z\,(3z-10)\,F(z)^4F'(z) + z^2F(z)^3F''(z)/2 + 6z^3\,(4z-3)\,F(z)^2F'(z)^3$$
$$- 8z^2\,(8z-3)\,F(z)^4F'(z)^2 + 32zF(z)^3F'(z) + 4z^2\,(4z-1)\,F(z)^6F''(z)$$
$$- 12z\,(5z-4)\,F(z)^5F'(z) - 2z^2\,(4z-3)\,F(z)^5F''(z) + 16z\,(3z-1)\,F(z)^6F'(z)$$
$$- 6zF(z)^2F'(z) + 42z^2F(z)^2F'(z)^2 + 12z^2\,(3z-4)\,F(z)^3F'(z)^2 - 8z^3\,(4z-1)\,F(z)^3F'(z)^3$$
$$+ 15z^3F(z)\,F'(z)^3 - 3z^2F(z)^4F''(z).$$

This would imply that $F(z)$ belongs to the class of *differentially algebraic* functions.

Ingredient 3

It turns out we can derive this equation from the implicit functional equations!

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$$B(z) = 2C(z(1 + B(z)))^2$$

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as formal power series.

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$$B(B^{\langle -1 \rangle}(z)) = 2C(B^{\langle -1 \rangle}(z)(1 + B(B^{\langle -1 \rangle}(z)))^2$$

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Substitute $z = B^{\langle -1 \rangle}(z)$.

$$C^{\langle -1 \rangle}\left(\frac{z}{2}\right) = (1+z)^2 B^{\langle -1 \rangle}(z)$$

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Together,

$$C^{\langle -1 \rangle}\left(\frac{z}{2}\right) = (1+z)^2 B^{\langle -1 \rangle}(z)$$

and

$$F^{\langle -1 \rangle}(z) = \frac{C^{\langle -1 \rangle}(z)}{(1+z)^2}$$

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Together,

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imply that

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Ingredient 3

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We just need to find a way to get the defining equation for the RHS from the known defining equation for $B(z)$.

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At this point, we can only rule this out empirically.

Ingredient 4

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Analytic Combinatorics 101:

- Treat a generating function as a complex-valued function. The asymptotic behavior of the coefficients is governed by the location and nature of the dominant singularities.

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Dominant singularity at

$$z \approx 0.07490791222594518 \pm 10^{-17}.$$

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$$\left(= -\frac{256}{9\pi^2} = -2.88202477915982994 31 \right)$$

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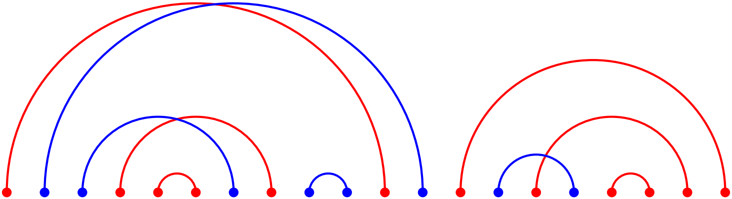
- Defining functional equations
- A polynomial-time algorithm to generate terms
- A proof of D-algebraicity
- Empirical evidence of asymptotics and non-D-finitness



Thank you!