Introduction:

The ratio between the circumference and the diameter of a circle - first denoted "π" in 1706 - has fascinated people throughout all of recorded history, likely because it represents such a fundamental geometric concept. In this talk, we'll look through some estimates of π, methods of finding these estimates, recent discoveries, and open questions regarding π.

Some Early Estimates for π:

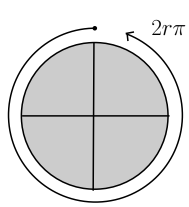
Perhaps the earliest estimate for π came from the Old Testament (translated around 550 BC):

> *And he made a molten sea, ten cubits from the one brim to the other: it was round all about, and his height was five cubits: and a line of thirty cubits did compass it round about.*
> \- I Kings 7:23

which effectively estimates π as 3. Of course, we shouldn't take this too seriously - **Rabbi Nehemiah** argued in his text Mishnat ha-Middot around 150 AD that the molten sea had an *outer* diameter of 10 cubits, and an *inner* circumference of 30 cubits. Estimating the width of the pool wall at about a "handbreadth", this brings the implied value of π much closer to the actual value. Whatever the true meaning of this passage, it hints at the fact that all ancient peoples were aware of the relationship between the diameter and circumference of a circle.

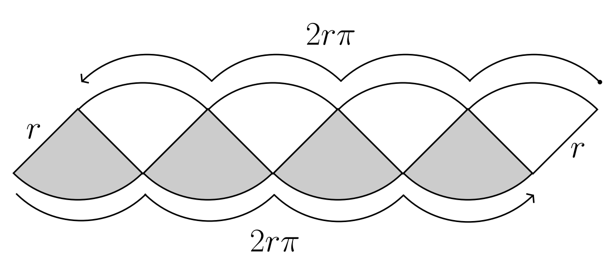
In fact, as early as 2000 BC, the Babylonians and the Egyptians already had reasonable estimates for π. While the method of estimation isn't known, they probably arose through construction of a circle with a fixed diameter, followed by measuring the circumference as accurately as possible. The Babylonians used the value $\pi = 3\frac{1}{8} = 3.125$, which was found on a tablet excavated in 1936 about 200 miles from Babylon. Meanwhile, the Egyptians used the value $\pi = \left(\frac{16}{9}\right)^2 = \frac{256}{81} \approx 3.1605\ldots$, which was written on the famous Ahmes Papyrus and discovered in an abandoned building in 1858.

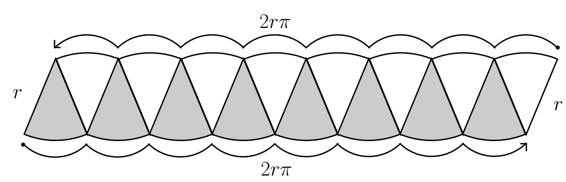
Despite only having estimations for the value of π, the ancients knew the formula for the area of a circle to be πr^2. Here is the most probable way they discovered this:



The circumference of a circle with radius r is $2r\pi$ by the definition of the constant π. We start with such a circle.

We take a second identical circle, split each up into four parts, and align them as shown. The length along the top and bottom is the circumference of the original circle.



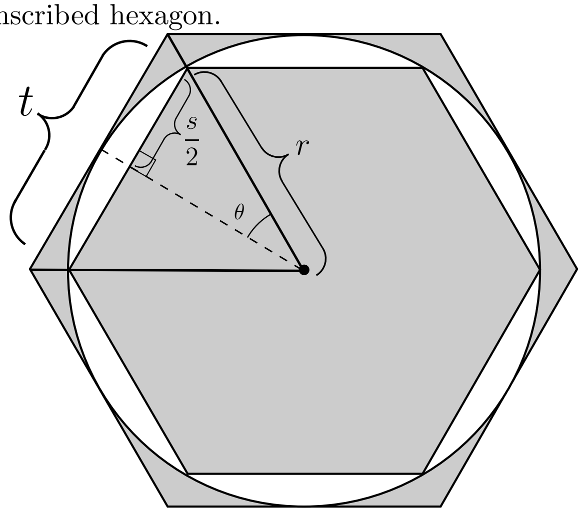


If we instead split each circle into eight equal parts and align them as shown, then the length along the top and bottom is still the circumference of the original circle, but the shape is closer to a rectangle.

Splitting the circle up into more and more parts, the resulting shape is closer and closer to a rectangle. This implies that the area of the two circles is $r \cdot 2r\pi = 2r^2\pi$, and hence the area of each circle is πr^2.

In the search for π's true value, **Archimedes** is thought to be the first to apply purely mathematical methods. In *On the Measurement of the Circle* he uses the following geometric construction to obtain bounds for π:

Construction by the Archimedean Method starts with a circle along with both an inscribed and circumscribed hexagon.



Observe that $s = 2r\sin\theta$ and $t = 2r\tan\theta$. Using $2\pi r$ as the circumference of the circle, and noting that the circumference of the circle lies between the circumferences of the inscribed and circumscribed polygons, we now have that

$$2rn\sin\theta < 2\pi r < 2rn\tan\theta, \qquad \text{and hence}$$
$$n\sin\theta < \pi < n\tan\theta.$$

In the hexagonal case, as above, we have $n = 6$ and $\theta = \frac{\pi}{6}$. As we double the number of sides k times, we have that:

$$2^k n \sin\frac{\theta}{2^k} < \pi < 2^k n \tan\frac{\theta}{2^k}.$$

As k is increases toward infinity, the bounds get closer to the actual value of π because the difference in circumference of the inscribed and circumscribed polygons gets closer to zero. **Archimedes** and others knew that for the starting hexagon, we have $\sin\theta = \frac{1}{2}$ and $\tan\theta = \frac{1}{\sqrt{3}}$. To increase k, they used various versions of the half-angle formulas.

For a millennium and a half, the Archimedean Method of estimating π reigned supreme. All that was needed to reach better and better estimates was the time and dedication to carry the computation further and further. Some of the estimates during this era are:

(380 AD) One of the Indian Siddhantas gives the value of π as $3\frac{177}{1250} = 3.1416$.

(450 AD) Chinese Mathematician **Tsu Chung-Chih** and his son **Tsu Keng-Chih** found

$$3.1415926 < \pi < 3.1415927.$$

It is likely that the Chinese were able to obtain such a high accuracy because they're knowledge of the digit 0 made them better equipped for calculation. An estimate of this level of accuracy would not be discovered in Europe for over 1000 years.

(598 AD) The Hindu mathematician **Brahmagupta** used $\pi = \sqrt{10} \approx 3.1623\ldots$. He likely observed that the perimeters of polygons with 12, 24, 48, and 96 sides inscribed in a circle of diameter 10, are given by the sequence $\sqrt{965}, \sqrt{981}, \sqrt{986}, \sqrt{987}$, and assumed that the sequence approached $\sqrt{1000} = 10\sqrt{10}$.

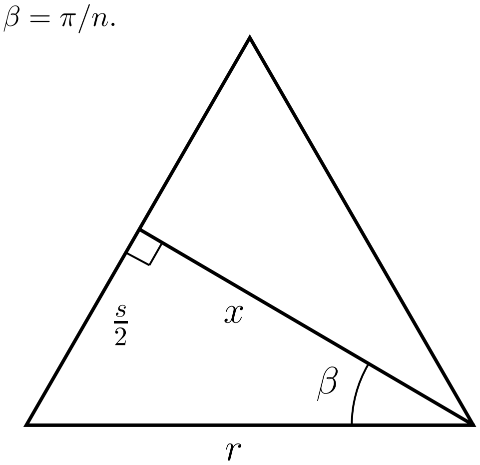
(1220 AD) **Leonardo of Pisa** - aka Fibonacci (literally, *Son of Bonaccio*) - used his era's new decimal arithmetic to obtain an estimation $\pi = \frac{846}{275} = 3.14\overline{18}$.

The New Age of π

François Viète:

After centuries of repeated applications of the Archimedean Method, the 16$^{\text{th}}$ century finally marked new progress. French mathematician **François Viète** (1540-1603) was primarily a lawyer, but did his most important mathematical work while in exile for his suspected sympathies toward the Protestant cause. It was in his famous *Analytical Art* that he coined some words we still use today, such as "negative", "coefficient", and "analytical". **Viète** used a method similar to the following to find the first ever analytical expression of π as a sequence of algebraic operations.

In the spirit of the Archimedean Method, **Viète** compared an n-gon to a $2n$-gon, but rather than comparing their perimeters, he compared their areas. Let the below diagram be one triangular section of the n-gon of radius r. Let 2β be the angle formed by this section, so that $\beta = \pi/n$.



Observe that the area of this triangle equals $\frac{1}{2}sx = (r\sin\beta)(r\cos\beta) = r^2\sin\beta\cos\beta$. Hence, the area of the n-gon is $nr^2\sin\beta\cos\beta$. Similarly, the area of the $2n$-gon is $2nr^2\sin\frac{\beta}{2}\cos\frac{\beta}{2} = nr^2\sin\beta$.

Hence, denoting $A(k)$ as the area of the k-gon, we have that:

$$\frac{A(n)}{A(2n)} = \frac{nr^2 \sin\beta\cos\beta}{nr^2\sin\beta} = \cos\beta.$$

Similarly,

$$\frac{A(n)}{A(4n)} = \frac{A(n)}{A(2n)}\frac{A(2n)}{A(4n)} = \cos\beta\cos\frac{\beta}{2}.$$

Repeating this process:

$$\frac{A(n)}{A(2^k n)} = \cos\beta\cos\frac{\beta}{2}\cdots\cos\frac{\beta}{2^k}.$$

Taking limits:

$$\lim_{k\to\infty}\frac{A(n)}{A(2^k n)} = \cos\beta\cos\frac{\beta}{2}\cos\frac{\beta}{4}\cos\frac{\beta}{8}\cdots.$$

Since $A(2^k n)$ approaches the area of the circle with radius r as $k \to \infty$, we have that

$$\frac{nr^2\cos\beta\sin\beta}{\pi r^2} = \cos\beta\cos\frac{\beta}{2}\cos\frac{\beta}{4}\cos\frac{\beta}{8}\cdots.$$

Hence

$$\pi = \frac{n\cos\beta\sin\beta}{\displaystyle\prod_{k=0}^{\infty}\cos\frac{\beta}{2^k}}.$$

Viète started with a square, setting $n = 4$ and $\beta = \pi/4$. Applying a half-angle formula

$$\cos\frac{\theta}{2} = \sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta},$$

our formula above yields

$$\pi = 2\cdot\frac{1}{\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdots}.$$

This convergence was proved formally by **Ferdinand Rudio** (1856-1929) in 1891.

This formula converges slowly, and so it isn't useful for calculating digits. **Viète** himself used the Archimedean Method to calculate π to 9 digits. Not many mathematicians wasted their time with the Archimedean Method after this calculation, principally because it served no practical purpose. Knowing π to 40 digits provides enough accuracy to compute the circumference of the Milky Way to within the radius of a proton!

John Wallis:

In 1655, British Mathematician **John Wallis** (1616-1703), considered the area under a circular arc - for which he had a formula due to **Descartes** - and used very tedious methods to derive

$$\pi = 2\cdot\frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdots\cdot}{3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdots\cdot}.$$

This was the first expression of π to have only rational quantities.

Lord Brouncker:

As discussed in class, British Mathematician **Lord William Brouncker** (1620-1684) used **Wallis'** formula to compute a general continued fraction for π:

$$\frac{\pi}{4} = \cfrac{1}{1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \cfrac{7^2}{\ddots}}}}}.$$

James Gregory:

The Scottish mathematician and astronomer **James Gregory** (1638-1675) made the next giant leap in the history of π. In 1671, he proved the equivalent of:

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots .$$

Setting $x = 1$, we have the formula

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right).$$

Though convergence is very slow (it takes over 300 terms to get two decimal places of accuracy), this is recognized as the first infinite series representation of π.

Isaac Newton:

The well-known British physicist, mathematician, astronomer, philosopher, and alchemist **Isaac Newton** could not resist spending some time working with π. He was aware of the Gregeory series, and knew that it converged far too slowly to be useful. Instead, he used the following derivation:

> Through his study of "fluxions" (derivatives) and "Flowing Quantities" (integrals), **Newton** knew the equivalent of
>
> $$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x).$$
>
> Using his own generalized binomial theorem, he also knew that
>
> $$\int \frac{dx}{\sqrt{1-x^2}} = \int \left(1 + \frac{1}{2}x^2 + \frac{1 \cdot 3}{2 \cdot 4}x^4 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots\right) dx.$$

Setting these two equal and integrating term-by-term, **Newton** found:

$$\arcsin(x) = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \cdots .$$

Substituting $x = \frac{1}{2}$:

$$\pi = 6\left(\frac{1}{2} + \frac{1}{2 \cdot 3 \cdot 2^2} + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5 \cdot 2^5} + \cdots\right).$$

This is another infinite series for π. However, it converges incredibly faster than the Gregory series. **Newton** himself used this method to calculate 15 correct decimal places for π, but later admitted in a letter "I am ashamed to tell you to how many figures I carried these computations, having no other business at the time."

John Machin:

Though he was a Professor of Astronomy at Gresham College in London, **John Machin** (1680-1750) is best known for his work with π, which came in 1706.

Let β be such that $\tan(\beta) = 1/5$. By the double angle formula:

$$\tan(2\beta) = \frac{2\tan(\beta)}{1 - \tan^2(\beta)} = \frac{2/5}{24/25} = \frac{5}{12}.$$

$$\tan(4\beta) = \frac{2\tan(2\beta)}{1 - \tan^2(2\beta)} = \frac{5/6}{119/144} = \frac{120}{119}.$$

We now notice that $\frac{120}{119}$ is just $\frac{1}{119}$ away from 1, which is $\tan\left(\frac{\pi}{4}\right)$. Using the formula for subtraction of angles inside of tangent:

$$\tan\left(4\beta - \frac{\pi}{4}\right) = \frac{\tan(4\beta) - \tan\left(\frac{\pi}{4}\right)}{1 + \tan(4\beta)\tan\left(\frac{\pi}{4}\right)} = \frac{\tan(4\beta) - 1}{1 + \tan(4\beta)} = \frac{1/119}{239/119} = \frac{1}{239}.$$

From this, we find the formula:

$$\arctan(1/239) = 4\beta - \frac{\pi}{4} = 4\arctan(1/5) - \frac{\pi}{4}.$$

Hence,

$$\frac{\pi}{4} = 4\arctan(1/5) - \arctan(1/239),$$

and thus,

$$\frac{\pi}{4} = 4\left(\frac{1}{5} - \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} - \cdots\right) - \left(\frac{1}{239} - \frac{1}{3 \cdot 239^3} + \frac{1}{5 \cdot 239^5} - \cdots\right).$$

This expansion is so useful because the terms in the first infinite sum are easy to calculate and the terms in the second sum converge extremely fast. Machin used the formula soon after discovering it to calculate π to 100 decimal places. In the very same year, the symbol "π" was first used by **William Jones** to denote the "periphery" of a circle of diameter 1.

Leonhard Euler:

It goes without saying that **Leonhard Euler** (1707 - 1783) found many formulas for π. It was one of these formulas that helped him make his name as a mathematician among the rest of Europe's mathematicians. At the age of 28, **Euler** solved the famous *Basel Problem*: finding the sum of the squares of the reciprocals of the natural numbers. Here's how he did it:

> Euler knew the formula
>
> $$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots .$$
>
> Solving for the equation $\sin(x) = 0$, we have roots $x = 0, \pm\pi, \pm 2\pi, \cdots$. So, making the assumption that we can treat this infinite polynomial like a finite polynomial, we can write $\sin(x)/x$ as:
>
> $$\frac{\sin(x)}{x} = \left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{2\pi}\right)\left(1 - \frac{x}{2\pi}\right)\cdots .$$
>
> Multiplying this out and collecting all terms with the coefficient x^2, we find that the x^2 term of $\sin(x)/x$ has coefficient:
>
> $$-\left(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + \cdots\right) = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}.$$
>
> Since the coefficient of x^2 in $\sin(x)/x$ is $-1/3! = -1/6$ as shown in the first formula (before factoring out an x), we conclude that:
>
> $$-\frac{1}{6} = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}, \quad \text{and hence:} \quad \frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

Johann Zacharias Dase:

Johann Dase (1824-1861) is the most well-known example of a "human calculator". Though he was completely unable to comprehend even basic mathematical theory, he could calculate large quantities in his head extremely rapidly. He used a formula derived similarly to Machin's Formula

$$\frac{\pi}{4} = \arctan\left(\frac{1}{2}\right) + \arctan\left(\frac{1}{5}\right) + \arctan\left(\frac{1}{8}\right)$$

to calculate 200 digits of π in his head, over the span of two months. After going on tour around Europe showcasing his ability, he calculated the log of the first million numbers to seven decimal places, created a table of the hyperbolic functions, and was halfway through factoring the numbers 7,000,000 - 10,000,000 when he died.

Johann Heinrich Lambert:

Despite all the attention π received over thousands years, it wasn't until 1767 it was proved that π was irrational, by **Johann Heinrich Lambert** (1728-1777). His theorem actually proved that if x is rational and nonzero, then $\tan(x)$ is irrational. Since $\tan\left(\frac{\pi}{4}\right) = 1$, which is rational, it follows that $\frac{\pi}{4}$ and hence π are irrational. In 1794, **Adrien-Marie Legendre** (1752-1833) made the proof more rigorous, and additionally proved that π^2 is irrational.

Carl Louis Ferdinand von Lindemann:

In 1882, **von Lindemann** (1852-1939) proved that π is a transcendental number, i.e., it is not the root of any polynomial with integer coefficients. **Lindemann** extended ideas from **Charles Hermite** (1822-1901) so that **Euler's Equation** $e^{i\pi} + 1 = 0$ implied the transcendence of $i\pi$. Since i is algebraic, this implied the transcendence of π.

<u>Circle Squarers:</u>

The proof of the transcendence of π finally shut the book on the ancient problem of "squaring the circle". The challenge posed by the ancient Greeks was to construct a square with the same area as a given circle in a finite number of steps using only a compass and ruler. It is shown in a typical graduate-level algebra class that any constructible lengths under these restraints must be algebraic of degree a power of two. If it was possible to square the circle, then the length π would be constructible. Since π is not algebraic, it is not possible to circle the square.

However, the fact that an act is impossible has never stopped those seeking fame and glory to try it, and this is no different. Even to this day, mathematics departments occasionally get proofs from those known as "circle-squarers" or "cyclometers" who claim to have successfully squared the circle.

In 1874, **John A. Parker** published <u>The Quadrature of the Circle</u>, in which he "proved" the value $\pi = \frac{20612}{6561} \approx 3.14159427\ldots$. This is closest to a value used by **Valentinus Otho** in 1573. The argument set forth by **Parker** amounted to the claim that the circumference should be measured as the length surrounding the outside of the circle, rather than the length of the boundary of the circle.

In 1897, the **Indiana State Legislature** almost passed a bill legislating an incorrect value of π. The construction of π and other geometric claims was a gift from **Edwin Goodwin**, a physician. In fact, the set of constructions gave two different values of π - the first was $\frac{4}{5/4} = 3.2$. The second, though likely accidental, was $\frac{16}{\sqrt{3}} \approx 9.2376\ldots$. Dr. Goodwin promised that with the passing of the bill, the state of Indiana would have free rights to the knowledge, while other states would have to pay royalties to publish his findings. The bill was directed to the *Committee on Swamp Lands* (no justification given in the legislative records), who then passed it on to the *Committee on Education*, which recommended its passage. The bill immediately passed the House unanimously and went to the Senate. Thankfully, on the day of the vote **Professor C.A. Waldo** was visiting the legislature on behalf of the Department of Mathematics of Purdue University on separate business. Upon hearing the bill read, he quickly intervened, giving the Senators a quick math lesson. The bill was tabled, and has not been revisited since.

Perhaps the most egregious example of a circle-squarer is **Carl Theodore Heisel**, who published <u>The Circle Squared Beyond Refutation</u>, in which he not only claims to square the circle, but also proves that decimals are inexact, disproves Pythagorean's Theorem as a special case, and shows the square roots of the first 100 positive integers to be rational numbers. His claim is $\pi = 3\frac{13}{81} = \frac{256}{81}$, which is the same value used in the <u>Ahmes Papyrus</u> over 4000 years ago. **Heisel** paid to have thousands of copies printed, and he distributed copies of his work to libraries and universities across the country, to aide in the education of mathematical students. **Heisel** did not think highly of mathematicians, saying in his preface:

> *As a class modern mathematicians do not seem to possess any doubt. They do not question anything. They blindly accept what is taught in the books as absolute and final. They learn their mathematics as a parrot learns language by simple imitation.*

The Computer Era

From nearly the instant that the first computer was turned on in the 1950s, programmers and computer enthusiasts have competed over who can calculate the most digits of π at the fastest speed. They do however owe a great deal to the mathematicians that continued to develop more efficient means of computing π. Many of the below algorithms use techniques such as the Fast Fourier Transform to help speed up arithmetic. The popular algorithm through the 1970s involved Machin's Formula at a high precision. Although this was not remarkably efficient, it was the best algorithm of the time.

Srinivasa Ramanujan:

Although **Srinivasa Ramanujan** (1887-1920) was not alive to see the computer era, much of his work was not discovered until this time period. **Ramanujan** had many absolutely astounding formulas for π. One of the most famous of these, discovered around 1910, is:

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801} \sum_{k=0}^{\infty} \frac{(4k)!(1103 + 26390k)}{(k!)^4 396^{4k}}.$$

This equation was used by American mathematician and programmer **Bill Gosper** (1943-) to calculate 17 million digits of π in 1985. The use of this equation yields an average of eight additional correct digits for each iteration of the sum. While this was certainly an advance, it was still only a linear algorithm.

Arithmetic-Geometric Mean:

The Arithmetic-Geometric Mean (AGM) iteration - made popular by **Gauss** (1777-1855) - provides a treasure trove of algorithmic methods to compute irrational numbers. In this section we develop the AGM, and in the next section we'll derive the Salamin-Brent Algorithm for computing digits of π with quadratic convergence.

> Between two numbers x and y, the arithmetic mean is $\dfrac{x+y}{2}$ and the geometric mean is $\sqrt{xy}$. To find the Arithmetic-Geometric Mean of x and y, denoted $\text{AGM}(x, y)$, we first set
>
> $$a_0 := \max(x, y), \qquad b_0 := \min(x, y).$$
>
> Now, we compute the sequences
>
> $$a_{n+1} := \frac{a_n + b_n}{2}, \qquad b_{n+1} := \sqrt{a_n b_n}.$$
>
> Remarkably, the sequences $\{a_n\}$ and $\{b_n\}$ not only converge but converge to the same limit - we call this limit the Arithmetic-Geometric Mean of a_0 and b_0. To see this, first recall that the geometric mean of two distinct numbers is always strictly less than their arithmetic mean. So:
>
> $$b_{n+1} = \sqrt{a_n b_n} \geq \sqrt{b_n^2} = b_n.$$
>
> Therefore, the sequence of geometric means is increasing. It is also bounded above by a_0, and therefore the sequence converges. Define $\text{AGM}(a_0, b_0) := \lim_{n \to \infty} b_n$. Now, note that
>
> $$\frac{b_{n+1}^2}{b_n} = \frac{\sqrt{a_n b_n}^2}{b_n} = \frac{a_n b_n}{b_n} = a_n.$$

Hence:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{b_{n+1}^2}{b_n} = \frac{\left(\lim\limits_{n\to\infty} b_{n+1}\right)^2}{\lim\limits_{n\to\infty} b_n} = \frac{(\mathrm{AGM}(a_0, b_0))^2}{\mathrm{AGM}(a_0, b_0)} = \mathrm{AGM}(a_0, b_0).$$

Thus, the sequences $\{a_n\}$ and $\{b_n\}$ both converge to the same limit. Since

$$a_{n+1} = \frac{a_n + b_n}{2} < \frac{a_n + a_n}{2} = a_n,$$

we have that

$$b_0 \le b_1 \le b_2 \le \cdots \le \mathrm{AGM}(a_0, b_0) \le \cdots \le a_2 \le a_1 \le a_0.$$

Salamin-Brent Algorithm:

In 1976, **Eugene Salamin** (no dates found) and **Richard Brent** (1946-) independently found an algorithm using the Arithmetic-Geometric Mean that could *quadratically* produce correct digits of π. Here is that algorithm:

Set $a_0 = 1$, $b_0 = \frac{1}{\sqrt{2}}$, and $s_0 = \frac{1}{2}$. For $k = 1, 2, 3, \ldots$, compute

$$a_k = \frac{a_{k-1} + b_{k-1}}{2},$$

$$b_k = \sqrt{a_{k-1} b_{k-1}},$$
$$c_k^2 = a_k^2 - b_k^2,$$
$$s_k = s_{k-1} - 2^k c_k^2,$$
$$p_k = \frac{2a_k^2}{s_k}.$$

Then, p_k converges quadratically to π. Successive iterations of this algorithm give $1, 4, 9, 20, 42, 85, 173, \ldots$ correct digits.

Of course, it is true that this algorithm requires the ability to perform high-precision square root operations. However, using a method called Newton Iteration, we can perform a square root in about the time it takes to perform three multiplications, making this algorithm must faster than the previous linear methods.

Below is the proof that the p_k sequence in the Salamin-Brent algorithm does converge to π. The fact that the convergence is quadratic can be found in **Salamin's** paper *Computation of* π *Using Arithmetic-Geometric Mean.*

Consider the elliptic integrals:

$$K(k) = \int_0^{\pi/2} \left(1 - k^2 \sin^2(t)\right)^{-1/2} dt,$$

$$E(k) = \int_0^{\pi/2} \left(1 - k^2 \sin^2(t)\right)^{1/2} dt.$$

If there exists k' such that $k^2 + k'^2 = 1$, then we can define two more elliptic integrals:

$$K'(k) := K(k') = K\left(\sqrt{1-k^2}\right),$$

$$E'(k) := E(k') = E\left(\sqrt{1-k^2}\right).$$

Now, we define symmetric versions of these last two:

$$\begin{aligned} I(a,b) &= \int_0^{\pi/2} \left(a^2\cos^2(t) + b^2\sin^2(t)\right)^{-1/2} dt \\ &= \frac{1}{a}\int_0^{\pi/2} \left(\cos^2(t) + \left(\frac{b}{a}\right)^2 \sin^2(t)\right)^{-1/2} dt \\ &= \frac{1}{a}\int_0^{\pi/2} \left(1 - \sin^2(t) + \left(\frac{b}{a}\right)^2 \sin^2(t)\right)^{-1/2} dt \\ &= \frac{1}{a}\int_0^{\pi/2} \left(1 - \left(1 - \frac{b^2}{a^2}\right)\sin^2(t)\right)^{-1/2} dt \\ &= a^{-1}K'(b/a). \end{aligned}$$

$$\begin{aligned} J(a,b) &= \int_0^{\pi/2} \left(a^2\cos^2(t) + b^2\sin^2(t)\right)^{1/2} dt \\ &= a\int_0^{\pi/2} \left(\cos^2(t) + \left(\frac{b}{a}\right)^2 \sin^2(t)\right)^{1/2} dt \\ &= a\int_0^{\pi/2} \left(1 - \sin^2(t) + \left(\frac{b}{a}\right)^2 \sin^2(t)\right)^{1/2} dt \\ &= a\int_0^{\pi/2} \left(1 - \left(1 - \frac{b^2}{a^2}\right)\sin^2(t)\right)^{1/2} dt \\ &= aE'(b/a). \end{aligned}$$

The transformations given by English mathematician **John Landen** (1719-1790) yield:

$$I(a_n, b_n) = I(a_{n+1}, b_{n+1}),$$

$$J(a_n, b_n) = 2J(a_{n+1}, b_{n+1}) - a_n b_n I(a_{n+1}, b_{n+1}).$$

So, letting $M := \mathrm{AGM}(a_0, b_0)$,

$$I(a_0, b_0) = I(M, M) = \frac{1}{M} = \frac{K'(1)}{M} = \frac{K(0)}{M} = \frac{\pi}{2M}.$$

In the *Handbook of Mathematical Functions*, by **Abramowitz** and **Stegun**, they derive

$$J(a_0, b_0) = \left(a_0^2 - \frac{1}{2}\sum_{j=0}^{\infty} 2^j c_j^2\right) I(a_0, b_0),$$

where $c_n^2 = a_n^2 - b_n^2$.

The **Legendre** Relation of elliptic integrals tells us that

$$K(k)E'(k) + K'(k)E(k) - K(k)K'(k) = \frac{\pi}{2}.$$

Making substitutions and multiplying through by aa', we have:

$$a^2 I(a,b)J(a',b') + (a')^2 I(a',b')J(a,b) - a^2(a')^2 I(a,b)I(a',b') = aa'\frac{\pi}{2}.$$

Now, set $a = a' = 1$ and $b = b' = 1/\sqrt{2}$. Using the relations above, and defining $M := \text{AGM}(1, 1/\sqrt{2})$, we see:

$$\frac{\pi}{2} = 2\left(\frac{\pi}{2M}\right)\left(\left(1 - \frac{1}{2}\sum_{j=0}^{\infty} 2^j c_j^2\right)\left(\frac{\pi}{2M}\right)\right) - \frac{\pi^2}{4M^2}$$

$$\frac{\pi}{2} = \left(\frac{\pi^2}{4M^2}\right)\left(2\left(1 - \frac{1}{2}\sum_{j=0}^{\infty} 2^j c_j^2\right) - 1\right)$$

$$\frac{2M^2}{\pi} = 2\left(1 - \frac{1}{2}\sum_{j=0}^{\infty} 2^j c_j^2\right) - 1$$

$$\pi = \frac{2M^2}{2\left(1 - \frac{1}{2}\sum_{j=0}^{\infty} 2^j c_j^2\right) - 1} = \frac{2M^2}{1 - \sum_{j=0}^{\infty} 2^j c_j^2} = \frac{2M^2}{\frac{1}{2} - \sum_{j=1}^{\infty} 2^j c_j^2}.$$

This formula matches the p_k terms of the algorithm as $k \to \infty$. Hence, the sequence $\{p_k\}$ converges to π. □

Jonathan and Peter Borwein:

The brothers **Jonathan Borwein** (1951-) and **Peter Borwein** (1953-) together in 1985 found methods similar to the Salamin-Brent Algorithm, but with faster convergence. The most notable version is the quartic:

Set $a_0 = 6 - 4\sqrt{2}$ and $y_0 = \sqrt{2} - 1$. For $k = 1, 2, 3, \ldots$, compute

$$y_{k+1} = \frac{1 - (1 - y_k^4)^{1/4}}{1 + (1 - y_k^4)^{1/4}},$$
$$a_{k+1} = a_k(1 + y_{k+1})^4 - 2^{2k+3} y_{k+1}(1 + y_{k+1} + y_{k+1}^2).$$

Then, p_k converges *quartically* to π.

The quartic algorithm was used by **Yasumasa Kanada** (no dates found) of the University of Tokyo to set the at-the-time record of 6.4 billion digits. He would later use this method to calculate up to 5 trillion digits.

It has since been shown that there are m^{th} order approximations for all m. In the paper Approximations to π via the Dedekind eta function (1996) by **J. Borwein** and **F. Garvan**, they give the following example of a nonic algorithm:

> Set $a_0 = \frac{1}{3}$, $r_0 = \frac{(\sqrt{3}-1)}{2}$, and $s_0 = (1 - r_0^3)^{1/3}$. For $k = 1, 2, 3, \ldots$, compute
>
> $$t = 1 + 2r_k$$
>
> $$u = [9r_k(1 + r_k + r_k^2)]^{1/3}$$
>
> $$v = t^2 + tu + u^2$$
>
> $$m = \frac{27(1 + s_k + s_k^2)}{v}$$
>
> $$a_{k+1} = ma_k + 3^{2k-1}(1 - m)$$
>
> $$s_{k+1} = \frac{(1 - r_k)^3}{(t + 2u)v}$$
>
> $$r_{k+1} = (1 - s_k^3)^{1/3}$$
>
> Now, $1/a_k$ converges *nonically* to π

It is worth noting that in terms of computer efficiency, the quartic algorithm is the most efficient of these. Though higher order algorithms converge faster, they are vastly more expensive in terms of computational time per stage.

The Chudnovsky Brothers:

In 1989, brothers **David Chudnovsky** (1947-) and **Gregory Chudnovsky** (1952-) discovered a formula for π in the spirit of the **Ramanujan** formula above:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k+3/2}}.$$

The **Chudnovsky Brothers** used this formula to calculate over 1 billion digits of π in the same year. Each iteration produces an average of 14 additional correct digits.

The current record of just over 10 trillion (decimal) digits, set in October 2011 by **Alexander Yee** and **Shigeru Kondo**, used this formula. The total computation time was 191 days on a very powerful desktop computer. The computation required 44 terabytes of disk space. (For comparison, it is often claimed that the Library of Congress contains roughly 10 terabytes of uncompressed print text.)

Rabinowitz-Wagon Algorithm:

In 1990, **Stanley Rabinowitz** (1947-) and **Stan Wagon** (1951-) developed a "spigot algorithm" for π. A "spigot algorithm" is one which can output the digits of π one at a time (in order only), but which does not use the previous digits as part of the computation of the new digits. However, an extraordinary amount of memory is needed to run the algorithm for a large number of digits. Because of this, it is never used for record-breaking attempts. It's attractiveness is in the fact that it uses only integer operations (additions, multiplication, reduction by a modulus), and therefore does not require high-precision floating point operations.

Computing Individual Digits of π:

After decades of finding and improving algorithms for computing π to high precision, all of which necessitated computing the first $d-1$ digits of π in order to compute the d^{th} digit, it came as a great surprise in 1996 that it was actually possible to compute individual hexadecimal (i.e., base 16) digits of π. This ability emerges from the amazing formula:

$$\pi = \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right).$$

The formula was found by **Simon Plouffe** (1956-) using an algorithm called *Pari-Gp* (slightly modified) that searches for integer relations for real numbers. This is similar to the more famous algorithm named *PSLQ*. After the formula was found, it was proved as follows:

It's clear that

$$\frac{x^{k-1}}{1-x^8} = \sum_{i=0}^{\infty} x^{k-1+8i}.$$

Now observe that

$$\int_0^{1/\sqrt{2}} \frac{x^{k-1}}{1-x^8}dx = \int_0^{1/\sqrt{2}} \sum_{i=0}^{\infty} x^{k-1+8i}dx = \sum_{i=0}^{\infty} \frac{x^{k+8i}}{8i+k}\Bigg|_0^{1/\sqrt{2}} = \frac{1}{2^{k/2}} \sum_{i=0}^{\infty} \frac{1}{16^i(8i+k)}.$$

Hence,

$$\sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right) = \int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8}$$

Substituting $y := \sqrt{2}x$:

$$\int_0^{1/\sqrt{2}} \frac{4\sqrt{2} - 8x^3 - 4\sqrt{2}x^4 - 8x^5}{1-x^8} = 16\int_0^1 \frac{4\sqrt{2} - 2\sqrt{2}y^3 - \sqrt{2}y^4 - \sqrt{2}y^5}{16-y^8}\frac{dy}{\sqrt{2}}$$

$$= 16\int_0^1 \frac{4-2y^3-y^4-y^5}{16-y^8}dy = 16\int_0^1 \frac{(y-1)(y^2+2)(y^2+2y+2)}{(y^2+2y+2)(y^2-2y+1)(y^2+2)(y^2-2)}dy$$

$$= 16\int_0^1 \frac{y-1}{y^4-2y^3+4y-4}dy.$$

Using partial fraction decomposition:

$$\frac{16y-16}{y^4-2y^3+4y-4} = \frac{4y}{y^2-2} - \frac{4y-8}{y^2-2y+2}.$$

So,

$$16\int_0^1 \frac{y-1}{y^4-2y^3+4y-4}dy = \int_0^1 \frac{4y}{y^2-2}dy - \int_0^1 \frac{4y-8}{y^2-2y+2}dy$$

$$= 2\ln(y^2-2)|_0^1 - 2\ln(y^2-2y+2)|_0^1 - 4\arctan(1-y)|_0^1$$

$$= 2(i\pi - (i\pi + \ln(2))) + 2\ln(2) - 4\left(0 - \frac{\pi}{4}\right) = \pi. \quad \square$$

The key value in this formula is the "16^i". This allows us to use the following method to compute the d^{th} hexadecimal digit of π:

Let

$$S := \sum_{i=0}^{\infty} \frac{1}{16^i}\left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6}\right).$$

To find the d^{th} hexadecimal digit of S, we can look at the first hexadecimal digit of the fractional part of $16^{d-1}S$ which will be denoted by $\text{frac}(16^{d-1}S)$.

Define

$$S_1 := \sum_{k=0}^{\infty} \frac{1}{16^k(8k+1)}, \qquad S_2 := \sum_{k=0}^{\infty} \frac{1}{16^k(8k+4)},$$
$$S_3 := \sum_{k=0}^{\infty} \frac{1}{16^k(8k+5)}, \qquad S_4 := \sum_{k=0}^{\infty} \frac{1}{16^k(8k+6)}.$$

so that

$$S = 4S_1 - 2S_2 - S_3 - S_4.$$

Note that for S_1:

$$\text{frac}(16^{d-1}S_1) = \text{frac}\left(\sum_{k=0}^{\infty} \frac{16^{d-1-k}}{8k+1}\right) = \text{frac}\left(\sum_{k=0}^{d-1} \frac{16^{d-1-k}}{8k+1} + \sum_{k=d}^{\infty} \frac{16^{d-1-k}}{8k+1}\right)$$

$$= \left[\sum_{k=0}^{d-1} \frac{16^{d-1-k} \bmod 8k+1}{8k+1} \bmod 1 \;+\; \sum_{k=d}^{\infty} \frac{16^{d-1-k}}{8k+1} \bmod 1\right] \bmod 1.$$

A computer can very rapidly calculate the term sum on the left via repeated squaring, modular reduction, and floating point division. Only a few terms need to be calculated of the sum on the right to prevent rounding errors. The result is a fraction between 0 and 1. Repeat this process with S_2, S_3, and S_4, then calculate S from these four quantities, reducing once more $\bmod 1$ if needed. Then, after converting to hexadecimal, the first digit of the remaining fractional part is the d^{th} hexadecimal digit of π.

This scheme is not significantly faster at calculating all digits of π, but it does allow the calculation of any specific digit in a much shorter time. In 1997, **Fabrice Bellard** (1972-) improved the algorithm to yield a ~43% increase in efficiency. An employee of "Yahoo!" used their distributed computational abilities to calculate the two quadrillionth binary bit of π - which is 0. The calculation took 23 days on over 1000 individual machines running in parallel.

At the time that this algorithm was published, the authors had no similar scheme that could compute the decimal digits of π. They were not able to find any simple formula with "10" in the place of "16". However, later in the same year (1996), **Plouffe** devised a somewhat unrelated and more complicated method of calculating an arbitrary decimal digit of π using low memory, but in $O(n^3 \log^3(n))$ time, which is highly inefficient. Again, **Bellard** refined the technique to obtain an $O(n^2)$ algorithm. Additional refinements have been made in that time, though finding a specific decimal digit still remains less efficient than the elegant process described above of finding a specific hexadecimal digit.

Open Questions

Normality:

Despite thousands of years of investigation, there remain many unsolved questions about the famous constant π. It took most of the last four millenium to discover that π is both irrational and transcendental. The next logical step is to determine the normality of π, i.e. whether each n-length string of digits occurs with limiting frequency b^{-n} in all bases b. All calculations of digits of π seem to suggest normality. However, in addition to the open question of whether π is normal in any particular base, it hasn't even been shown that any particular digit repeats infinitely often!

The digit-extraction algorithms above have provided a new avenue for recent investigation. It is thought that the normality of π is equivalent to a "plausible" conjecture in the field of chaos theory.

Algebraic Independence:

Another famous open question is whether e and π are algebraically independent. It was shown that $\{\pi, e^{\pi}, \Gamma(\frac{1}{4})\}$ is an algebraically independent set over $\mathbb{Q}$ in 1996 by **Yuri Nesterenko** (1946-).

Irrationality Measure:

The irrationality measure of a real number gives a sense of how closely the number can be approximated by rationals. For example, algebraic numbers all have irrationality measure 2 (**Roth**, 1955). However, there are transcendental numbers that also have irrationality type 2: for example, the constant e.

The first irrationality measure of π was given by **Mahler** in 1953:

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{42}}, \quad q \geq 2.$$

In 1974, **Mignotte** improved this:

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{20.6}}, \quad q \geq 2.$$

The **Chudnovsky Brothers** mentioned above refined this to:

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{14.65}}, \quad q \text{ large}.$$

The current best measure was found by **Salikhov** in 2008:

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{7.6063\ldots}}, \quad q \text{ large}.$$

However, in April **Alekseyev** showed that if the Flint Hills series

$$\sum_{n=1}^{\infty} \frac{\csc^2(n)}{n^3}$$

converges, then this implies an irrationality measure

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^{2.5}},$$

which is remarkably better than the current estimate. The Flint Hills series appears to converges upon visual inspection, but this convergence has not been proven. The difficulty is that $\csc^2(n)$ can sporadically take very large values.

When considering the intricate and vast history of π - from the Ancient Babylonians and Egyptians all the way through the computer age - it is not unthinkable that it could be hundreds if not thousands of years before these open questions are finally settled.