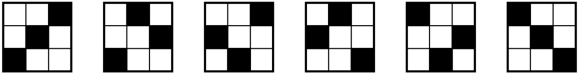
Equipopularity in the Separable Permutations

(joint work with Cheyne Homberger and Michael Albert)

Jay Pantone

University of Florida



AMS Section Meeting – Eau Claire, WI

September 20, 2014

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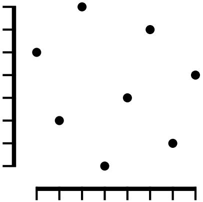

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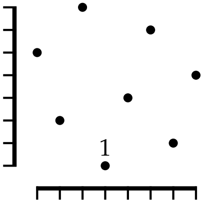

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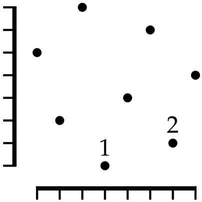

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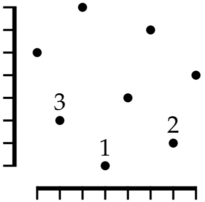

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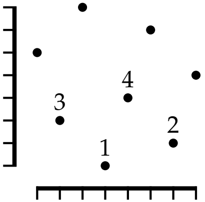

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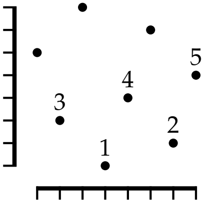

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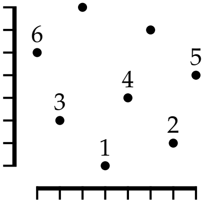

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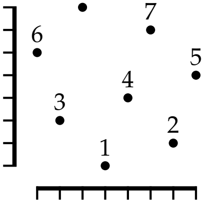
63814725 $\Longleftrightarrow$ 

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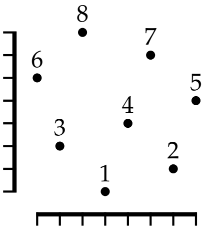

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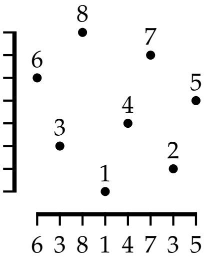

Pattern Avoidance

A permutation π *contains* a permutation σ, denoted $\sigma \leq \pi$, if there are entries of π which occur in the same relative order as the entries of σ.

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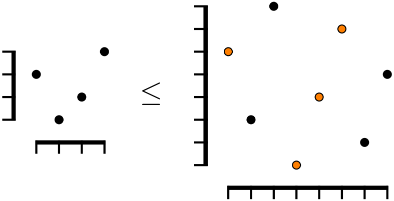

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A permutation class can be specified by the unique minimal set of permutations which it does not contain, called its *basis*. A permutation class with basis B is denoted $\mathrm{Av}(B)$.

Popularity of Patterns

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Two patterns σ and τ are said to be *equipopular* in a class $\mathcal{C}$ if

$$\text{pop}_{\mathcal{C}_n}(\sigma) = \text{pop}_{\mathcal{C}_n}(\tau)$$

for all $n \geq 1$. We denote this by $\sigma \overset{\mathcal{C}}{\sim} \tau$ (or just $\sigma \sim \tau$ when the context is clear).

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We say that σ and τ are in the same *equipopularity class*.

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In proper classes, the story is much different.

Known Results

Bóna initially showed that in $\mathrm{Av}(132)$,

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We will work with a class that properly contains Av(132): the class of *separable permutations*.

THE SEPARABLE PERMUTATIONS

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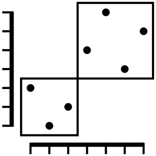
This class has a very nice structural decomposition in terms of *sums* and *skew sums*.

Sum and Skew Decomposable Permutations

A permutation that be split into a permutation τ above and to the right of a permutation σ is said to be *sum decomposable*, and can also be denoted $\sigma \oplus \tau$.



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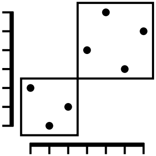

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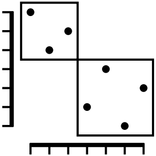

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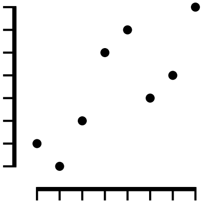

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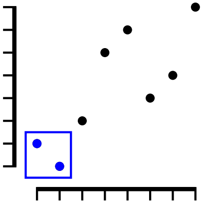

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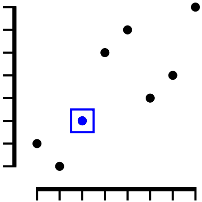

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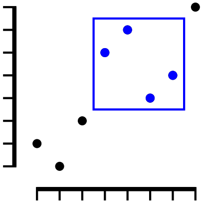

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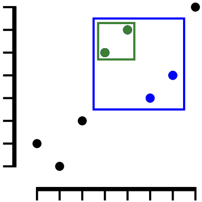

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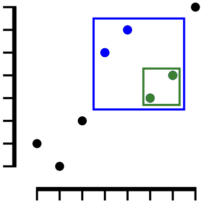

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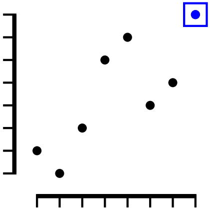

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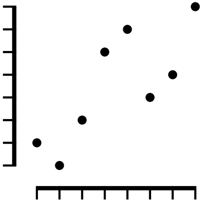

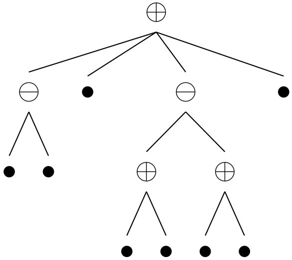

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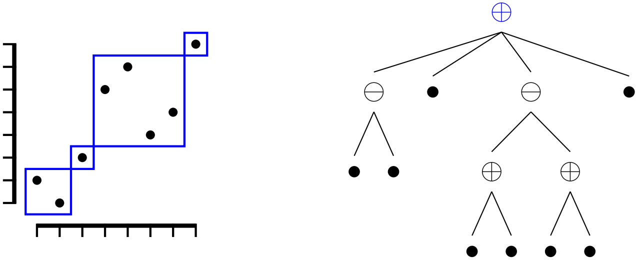

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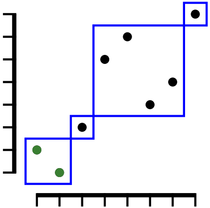

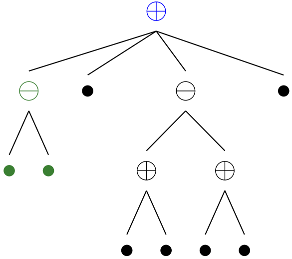

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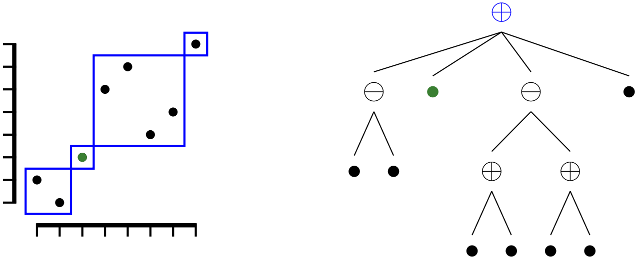

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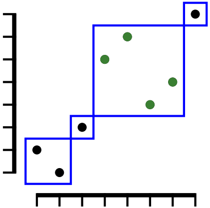

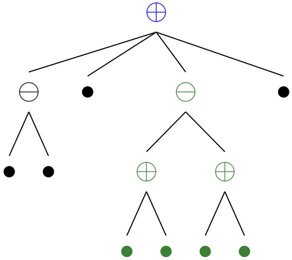

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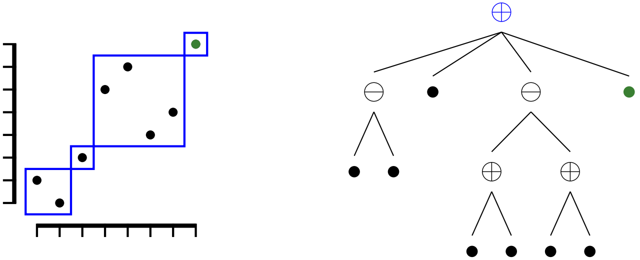

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It turns out that each equipopularity class of permutations of length n can be represented by a unique integer partition of $n-1$.

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Each internal node with d children contributes a part of $d-1$ to the partition.

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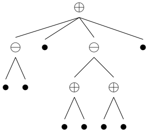
$21367458 =$ 

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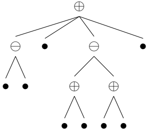

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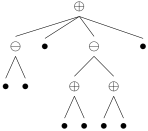
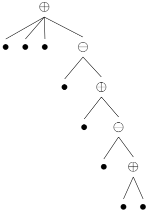
$21367458 =$  $\longrightarrow \quad 3+1+1+1+1.$

$12384756 =$ 

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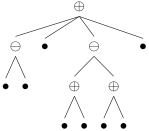
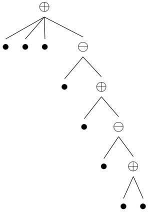


$\longrightarrow \quad 3+1+1+1+1.$

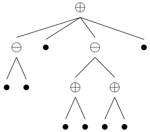
$12384756 =$

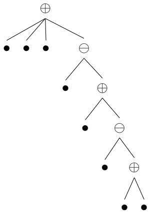


$\longrightarrow \quad 3+1+1+1+1.$

Equipopularity Classes of the Separables

Examples:

$21367458 =$  $\longrightarrow \quad 3+1+1+1+1.$

$12384756 =$  $\longrightarrow \quad 3+1+1+1+1.$

These two permutations correspond to the same partition, and thus are equipopular among the separable permutations.

POPULARITY-PRESERVING OPERATIONS

The eight symmetries of a permutation are generated by the operations of reversal and group-theoretical inverse.

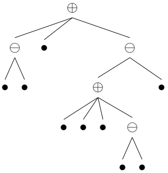
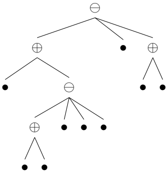
Popularity-Preserving Operations

The eight symmetries of a permutation are generated by the operations of reversal and group-theoretical inverse.

The separable permutations are closed under all symmetries, so if π_1 and π_2 are symmetries of each other, then they are automatically equipopular.

Popularity-Preserving Operations

Reversal corresponds to switching all $\oplus$ and $\ominus$ labels and reversing the children of each node.

213567984 =  $\longleftrightarrow$  = 489765312

Popularity-Preserving Operations

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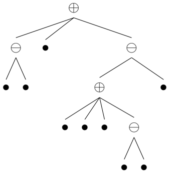
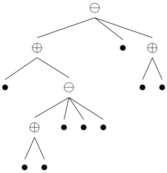
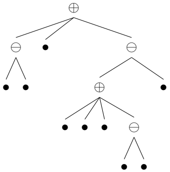
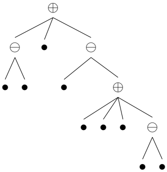
$213567984 =$  $\longleftrightarrow$  $= 489765312$

Inversion corresponds to reversing the children of all $\ominus$-labeled nodes.

$213567984 =$  $\longleftrightarrow$  $= 213945687$

Popularity-Preserving Operations

There are two non-trivial operations that preserve popularity: *rearrangement of children* and *exchange of children*.

Rearrangement of Children

The children of any node can be rearranged and equipopularity is preserved.

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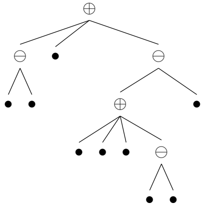
213567984



$\sim$

Rearrangement of Children

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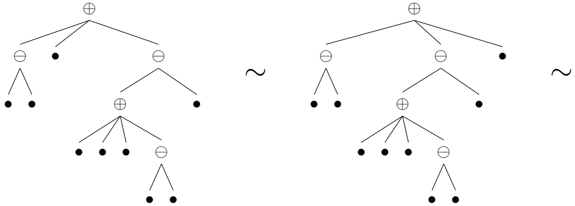
213567984 214568739



REARRANGEMENT OF CHILDREN

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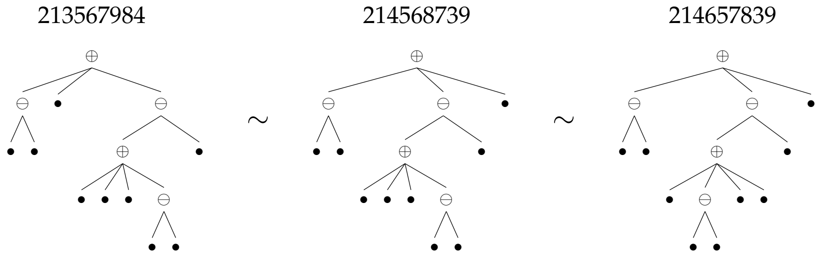
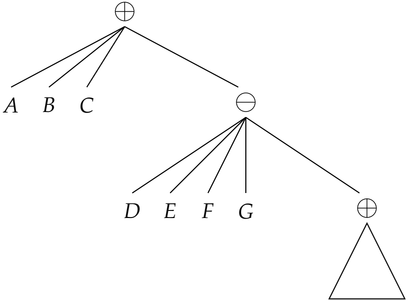
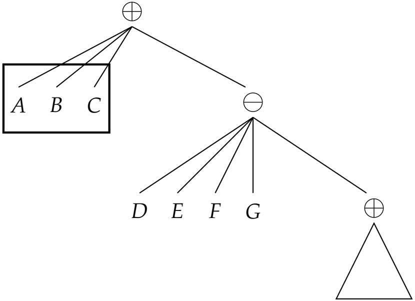


Exchange of Children



Exchange of Children



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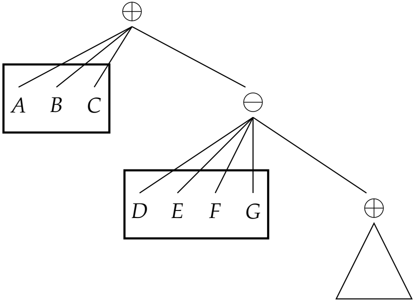

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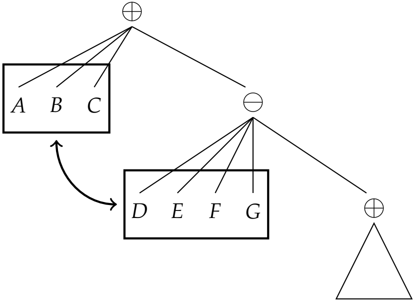
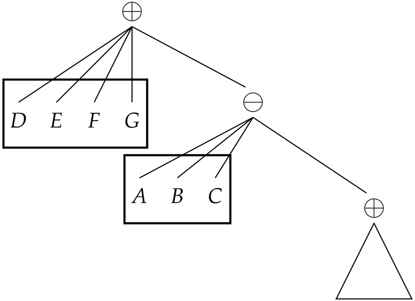


Exchange of Children



Canonical Form

Using the two symmetry operations (reversal, inversion) and the two popularity-preserving operations (rearrangement of children, exchange of children), every separable tree can be turned into "canonical form".

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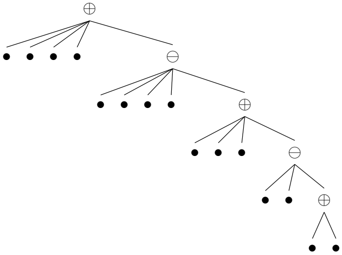

$$4 + 4 + 3 + 2 + 1$$

Surjection → Bijection?

Hence there is a (well-defined) surjection from equipopularity classes of permutations of length n to integer partitions of $n - 1$.

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As with previous similar problems, this part is harder.

FACTORING WEDGES

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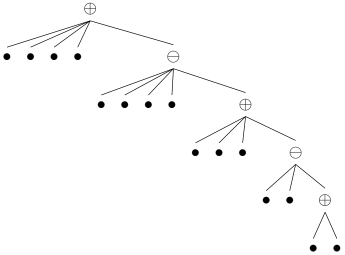

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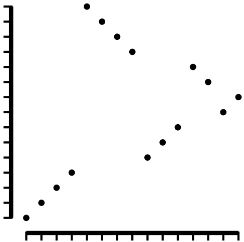

Factoring Wedges

Using a structural description of the separable permutations and some analytic combinatorics, we show that the popularity generating function for a permutation represented by the partition $m_1 + m_2 + \cdots + m_k$ can be factored into generating functions

$$F_{m_1} F_{m_2} \cdots F_{m_k},$$

where F_ℓ is the popularity generating function for the monotone permutation of length ℓ.

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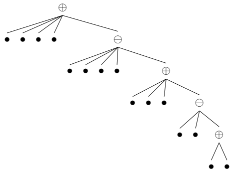

$$\longrightarrow \quad F_4^2 F_3 F_2 F_1$$

Factoring Wedges

So, if we can look at a product

$$\prod_{i=1}^{k} F_{m_i}$$

and identify the F_{m_i} (and hence the m_i), this will show that each "canonical" separable permutation has a different popularity.

FACTORING WEDGES

So, if we can look at a product

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and identify the F_{m_i} (and hence the m_i), this will show that each "canonical" separable permutation has a different popularity.

This is where we got stuck. It's easy enough to compute each individual $F_1, F_2, \ldots$ and observe that each F_m has a root that none of the others do (and hence they can be identified from a product), but we didn't know how to show this property in general.

And then...

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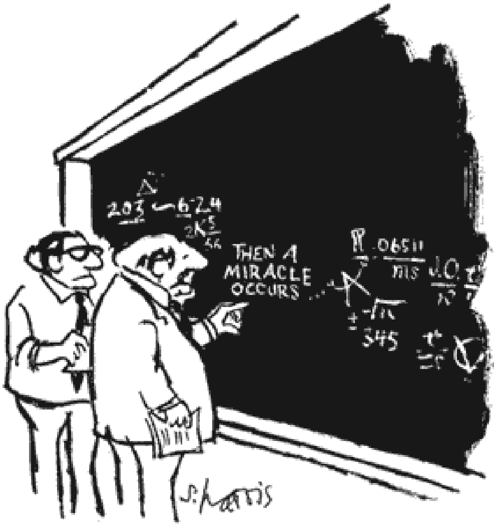

The Miracle

It turns out that the F_m are a close-variant of the *Gegenbauer polynomials*, which is a family of orthogonal polynomials. (The Narayana numbers are also involved... we can't explain this.)

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THE MIRACLE

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It follows immediately that we can factor a product of the F_m by looking at the roots of the product.

Thus our surjection is actually a bijection, and we've proved that two separable permutations are equipopular if and only if they correspond to the same integer partition.

Applications to Previous Work

The same orthogonal polynomial trick works for the Av(132) case, greatly simplifying the paper of Chua and Sankar which showed that Rudolph's surjection was really a bijection.

APPLICATIONS TO PREVIOUS WORK

The same orthogonal polynomial trick works for the Av(132) case, greatly simplifying the paper of Chua and Sankar which showed that Rudolph's surjection was really a bijection.

We think a small tweak to the first part would also provide popularity-preserving operations for Av(132).

FUTURE WORK

We would love to find an explanation for why the Narayana numbers pop up, but they arise in a strange way.

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The class of separable permutations is the smallest substitution-closed class. Can we apply some of these methods to other substitution-closed classes? (The problem: now there are simple permutations.)

Thanks for coming! Any questions?