The Method of Differential Approximants

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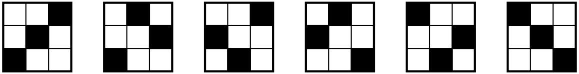

Loyola University, Chicago IL

October 3, 2015

Classification of Generating Functions

A GF $F(x)$ is *holonomic* or *differentially finite* (*D-finite*) if there is a linear differential equation with polynomial (in x) coefficients that has $F(x)$ as a solution.

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$$\implies F(x) = [\textit{messy}]$$

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Empirically: a lot.

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Rigorously: nothing.
Empirically: a lot.

Disclaimer: you can construct pathological examples that break almost everything.

The Ratio Method

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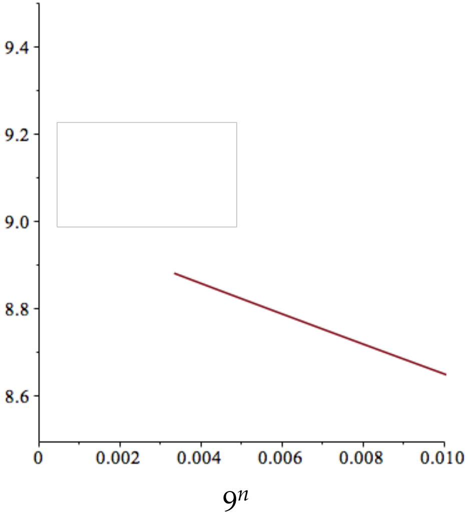
So, if we plot the set of points $\left(\dfrac{1}{n},\ r_n\right)$, we expect a curve with y-intercept μ and slope $\mu\gamma$.

THE RATIO METHOD



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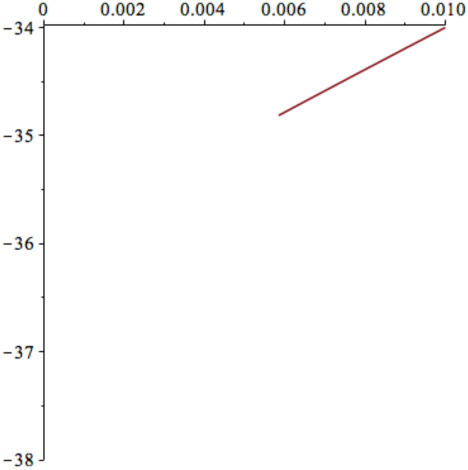

$9^n n^{-4}$

Singularity of Solutions to LODEs

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It is possible to derive (almost) full asymptotic expansions *without actually solving the ODE*.

LOCATION OF SINGULARITIES

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$$+x^2(9x-1)(x-1)F''(x)$$
$$\implies \text{possible singularities at } x = 0,\ \frac{1}{9},\ 1.$$

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The roots of the indicial equation are the smallest powers of $(x - T)$ in each power series expansion of $F(x)$ around $x = T$.

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Start with:

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$$0 = -36(2x-1)F(x)-2(180x^2-146x+18)F'(x)-4x(63x^2-62x+7)F''(x)-4x^2(9x-1)(x-1)F'''(x)$$

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Substitute $F(x) = \left(x - \frac{1}{9}\right)^r$:

$$0 = \frac{32}{81}r(r-1)(r-3)\left(x - \frac{1}{9}\right)^{r-2} + O\left(\left(x - \frac{1}{9}\right)^{r-1}\right)$$

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Thus there the power series solutions have lowest order terms $1, x - \frac{1}{9}$ and $\left(x - \frac{1}{9}\right)^3$.

$$C_1 + \cdots, \qquad C_2\left(x - \frac{1}{9}\right) + \cdots, \qquad C_3\left(x - \frac{1}{9}\right)^3 + \cdots$$

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These three power series form a basis of solutions to the ODE.

TAUBERIAN THEORY (AKA, THE STANDARD FUNCTION SCALE)

$$F(x) \underset{x \to 1}{\sim} \left(\frac{1}{1-x}\right)^{\alpha} \left(\frac{1}{x} \log\left(\frac{1}{1-x}\right)\right)^{\beta}$$

$$\Longleftrightarrow$$

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with some restrictions... (e.g., for $\alpha \in \mathbb{Z}_{<0}$, no $\Gamma(\alpha)$ term)

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Our solution:

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Really, the first two terms are extraneous, leaving

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Asymptotic analysis is performed on all of these, and "averaged out" to give a prediction of the actual asymptotic behavior of the underlying sequence.

Fitting to D-finite Form

Assume a total differential order and a degree of each polynomial coefficient:

$$(a_{00}+a_{01}x+a_{02}x^2)+(a_{10}+a_{11}x)F(x)+(a_{20}+a_{21}x+a_{22}x^2)F'(x)$$
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Since there are 10 unknowns and order 2, substitute $10+2=12$ known terms:

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After doing the algebra, set the coefficient of x^n of the LHS equal to zero for $n = 0, 1, \ldots, 9$, and solve the linear system.

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Repeat this with a lot of differential approximants of the same order, varying the degrees of the polynomial coefficients slightly.

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Given: N initial terms

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- For each such singularity, compute the exponents in each approximant, drop $Y\%$ of outliers, return the remaining interval

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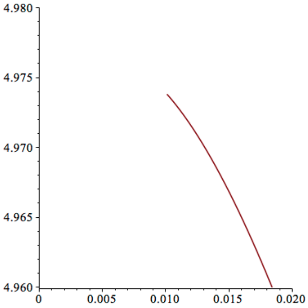

AV(4123,4231)

Dominant singularity at

$$x = 0.20092861430290850630066749465511761874842136315274588937508575345652139 \pm 10^{-71}$$

with critical exponent

$$-0.999999999999999999999999999999999999999999999999999999999999999999980 \pm 10^{-67}$$

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Other singularities:

- $x = 0.2061466445892927115969855884 0 \pm 10^{-29}$
- $x = 0.20724531832263 \pm 10^{-14}$
- $x = 0.24870945696 \pm 0.00390217832i \pm (1+i)10^{-11}$

AV(4123,4312)

Dominant singularity at

$$x = 0.1715728752538099023966225515806038428606562492461038536466405240185\ldots$$
$$\ldots 3504307578592229922493134471685452997230753817540595015032 \pm 10^{-125} = 3 - 2\sqrt{2}$$

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$$0.500000000000000000000000000000000000000000000000000000000000000000000000\ldots$$
$$\ldots 0000000000000000000000000000000000000000000000040 \pm 10^{-119}$$

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Much better than the ratio method.

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Other singularities:

- $x = 0.1597726221747308831 \pm 10^{-19}$
- $x = 0.2439516153417787 \pm 0.1200274949895230i \pm (1+i)10^{-16}$
- $x = 0.643104131 \pm 10^{-9}$

Av(4231,4321)

Dominant singularity at

$$x = 0.16970755392927711099361272283380722225065601529439005733\ldots$$
$$\ldots 3192350433804002842072267740380 \pm 10^{-93}$$

with critical exponent

$$-1.0000000000000000000000000000000000000000000000000000000000\ldots$$
$$\ldots 0000000000000000000000000000020 \pm 10^{-64}$$

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$$\ldots 0000000000000000000000000000020 \pm 10^{-64}$$

Other singularities:

- $x = .1715728752538099023966225515806038 42\ldots \pm 10^{-70}$
- $x = 0.167216154\ldots \pm 0.004184699\ldots i \pm (1+i)10^{-37}$
- $x = 0.16666666666666666666666666666666666 \pm 10^{-37}$
- $x = 0.1458980337503154 5 \pm 10^{-17}$
- $x = 0.241270585132 \pm 0.024119881572i \pm (1+i)10^{-12}$
- $x = 0.15757012628 \pm 0.04621114592i \pm (1+i)10^{-11}$
- $\ldots$

AV(4123,4231,4312)

Dominant singularity at

$$x = 0.22379330134707544472559413080653078228254418356103196904827043215739850630\ldots$$
$$\ldots 9150301940112196901230072474070042179174383297 \pm 10^{-120}$$

with critical exponent

$$-1.00000000000000000000000000000000000000000000000000000000000000000000000000000\ldots$$
$$\ldots 0000000000000000000000000000000000000000 \pm 10^{-115}$$

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$$\ldots 9150301940112196901230072474070042179174383297 \pm 10^{-120}$$

with critical exponent

$$-1.00000000000000000000000000000000000000000000000000000000000000000000000000000\ldots$$
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Other singularities:

- $x = 0.238063759808028162762342330112220060\ldots \pm 10^{-64}$
- $x = 0.2425743178578198539792244185837648 6\ldots \pm 10^{-38}$
- $x = 0.244744854382140527597172 \pm 10^{-24}$
- $x = 0.24600369400712 \pm 10^{-14}$

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In some cases, you can use the differential approximants to guess the next "few" terms in the sequence, and then plug these back into the ratio method to get a better estimation.

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- Better description of asymptotics (including logarithmic terms)

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- Better code (coming soon!)

Thanks for coming!