The Method of Differential Approximation in Enumerative Combinatorics

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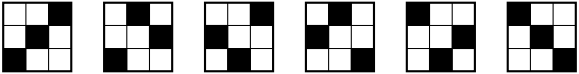

SIAM Applied Algebraic Geometry
Symbolic Combinatorics

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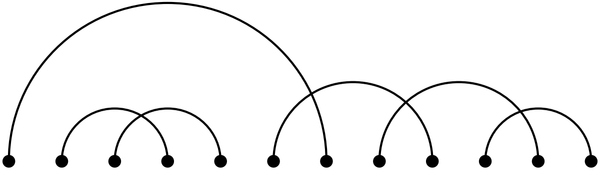
What can you do?

Running Example: 2-Colorable Matchings



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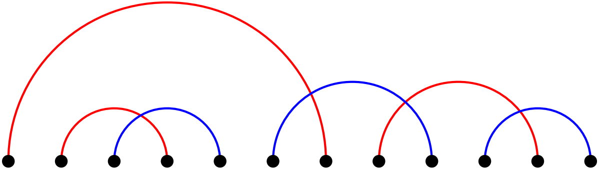

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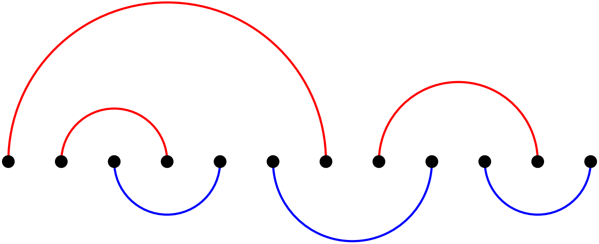

2-Colorable Matchings

By brute force, we can compute some initial terms of the counting sequence:

$$F(z) = 1 + z + 3z^2 + 14z^3 + 84z^4 + 592z^5 + 4659z^6 + \cdots .$$

2-Colorable Matchings

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Two experimental algorithms:

1. Automated Guessing
2. Differential Approximation

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- It can point you in the right direction.
- If nothing else, you have a cool conjecture.

The Hierarchy of Generating Functions

A generating function is *D-finite* if it is the solution to a linear differential equation.

$$p_k(z)F^{(k)}(z) + \cdots + p_1(z)F'(z) + p_0(z)F(z) = 0.$$

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Example:

$$z^3 - zF(z) + 5F(z)^2 - 8zF(z)F'(z) + 3z^2(F'(z))^2 = 0$$

2-Colorable Matchings

Conjecture:

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$$(-8+12\,z)\,F(z)^7+(-12\,z+20)\,F(z)^6+(3\,z-18)\,F(z)^5-F(z)^3-8\,z^3F'(z)^3+7\,F(z)^4$$
$$-12\,z^2F\,(z)\,F'(z)^2+6\,z\,(3\,z-10)\,F(z)^4F'(z)+1/2\,z^2F(z)^3F''(z)+6\,z^3\,(4\,z-3)\,F(z)^2F'(z)^3$$
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Great... now what?

Differential Approximation

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My goals:

- More accurate predictions.
- More types of predictions.
- Rigorous footing.
- Fast implementation.

Analytic Combinatorics

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Example: If $f(z) \sim (1-3z)^{-4/7}$ as $z \to \frac{1}{3}$, then

$$a_n \sim \frac{1}{\Gamma(4/7)} 3^n n^{-3/7}$$

(as long as $\frac{1}{3}$ is the unique singularity closest to the origin).

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$$\left(345z^2 - 167z + 18\right) F'(z) + \left(-174z + 65\right) F(z) + (43z^2 + 204z - 83) = 0$$

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The singularities of a D-finite model for a_n approximate the singularities of the generating function of a_n.

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Two questions:

- Given a D-finite model, how do we study the local behavior near singularities?
- How do we coalesce the information from each D-finite model into a single approximation?

Local Behavior of D-finite Functions

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$$F(z) \sim (1-\mu z)^{\alpha} \log(1-\mu z)^{\beta} H(z),$$

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Leads to asymptotic behavior:

$$a_n \sim C\mu^n n^{-\alpha-1} \log(n)^{(\beta \text{ or } \beta-1)}.$$

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Suppose

$$p_k(z)f^{(k)}(z) + p_{k-1}(z)f^{(k-1)}(z) + \cdots + p_1(z)f'(z) + p_0(z)f(z) = 0.$$

Every singularity of $f(z)$ is a root of $p_k(z)$. These are the candidates for $1/\mu$.

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Solve $p_k(z)$ to find candidates for μ.

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It's an *iterative process*.

Aggregation

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Asymptotics of D-finite models constructed using 40 terms:

- $a_n \approx C \cdot (2.99898)^n n^{-2.47709}$
- $a_n \approx C \cdot (2.99912)^n n^{-2.48602}$
- $\cdots$

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How does it do this?

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Step 3: For logarithmic terms, procedure still unclear.

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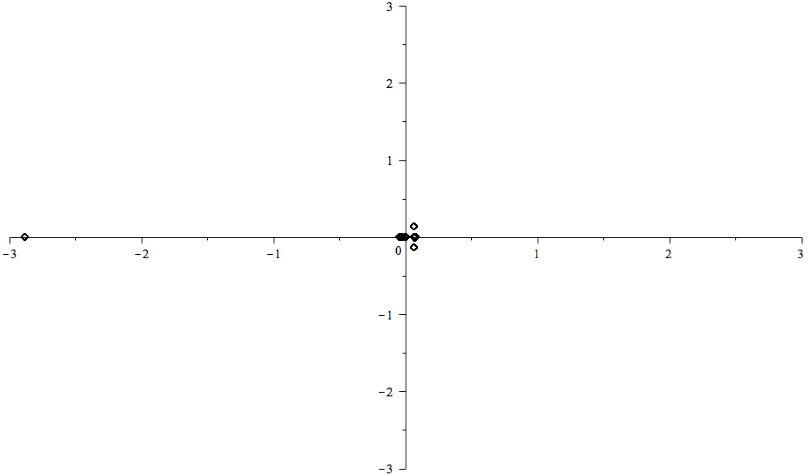

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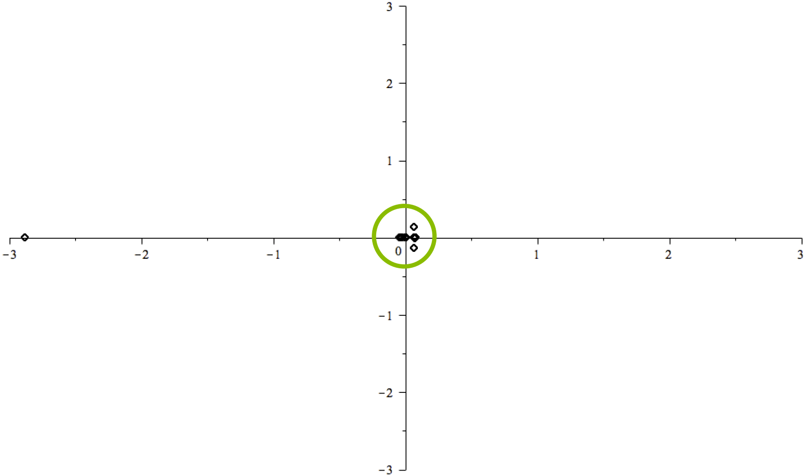

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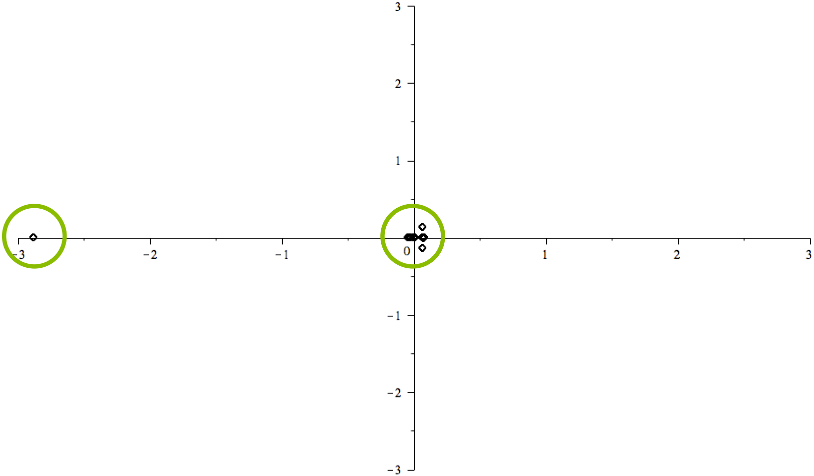

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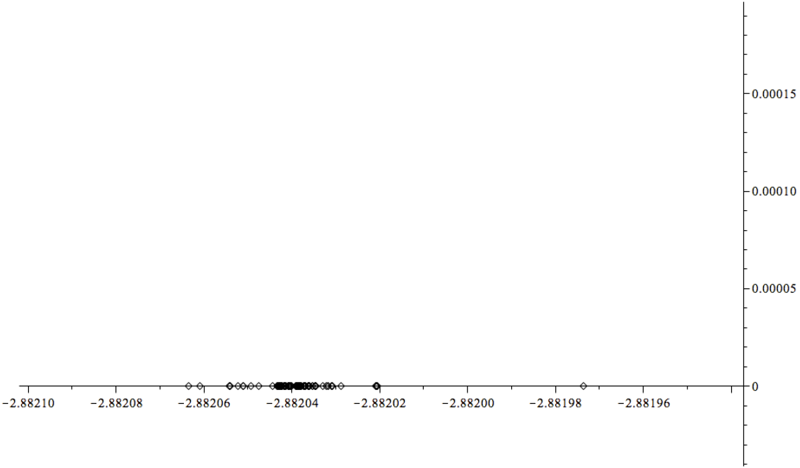

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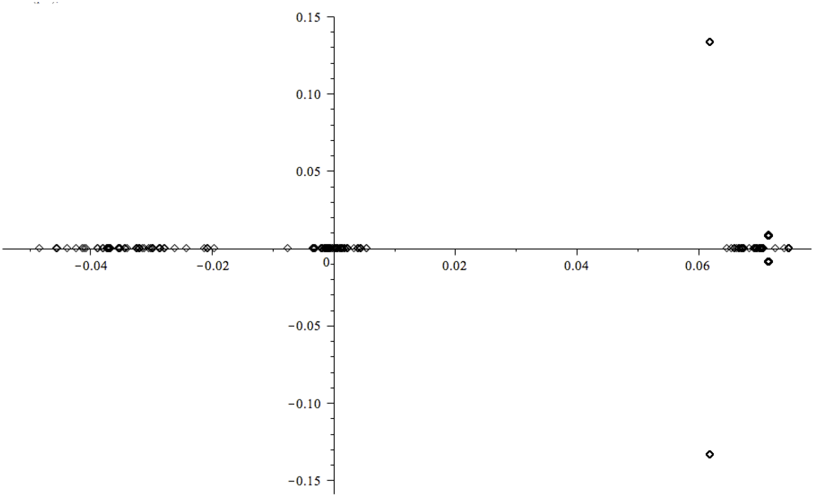

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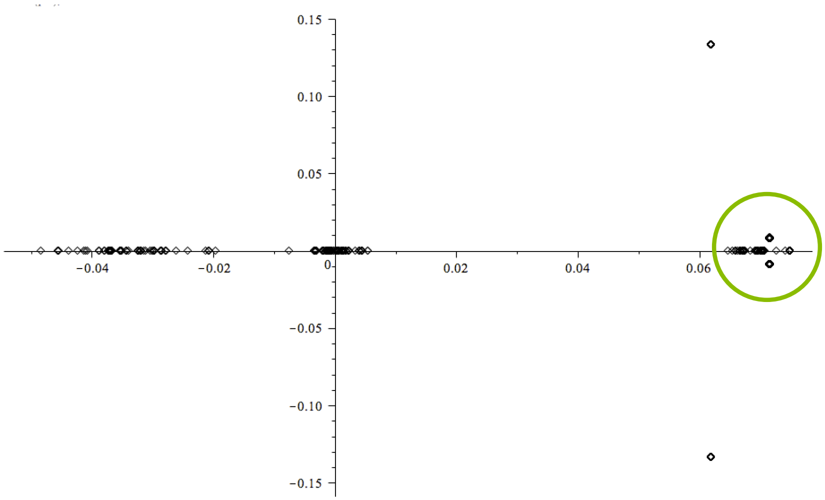

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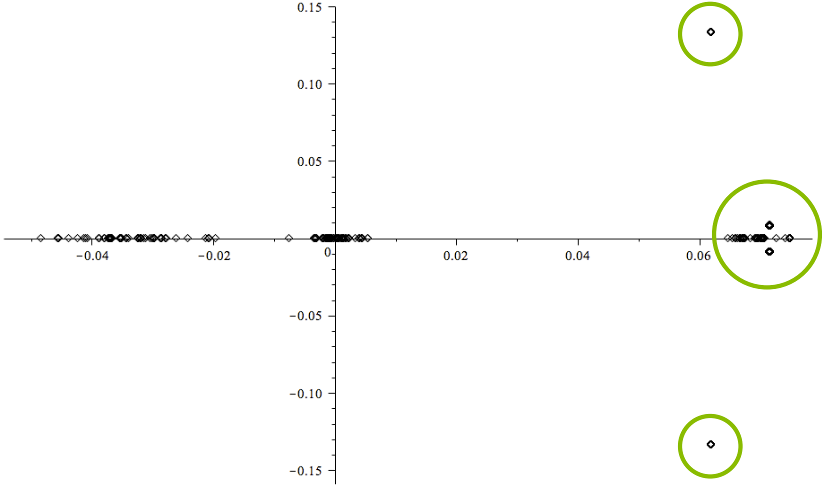

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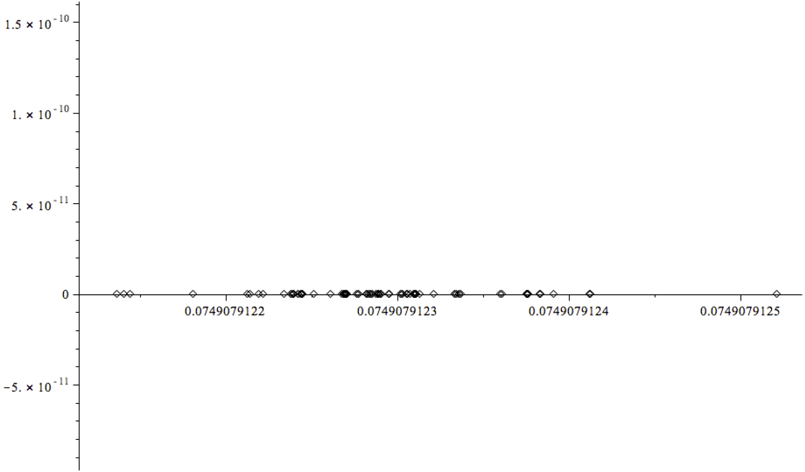

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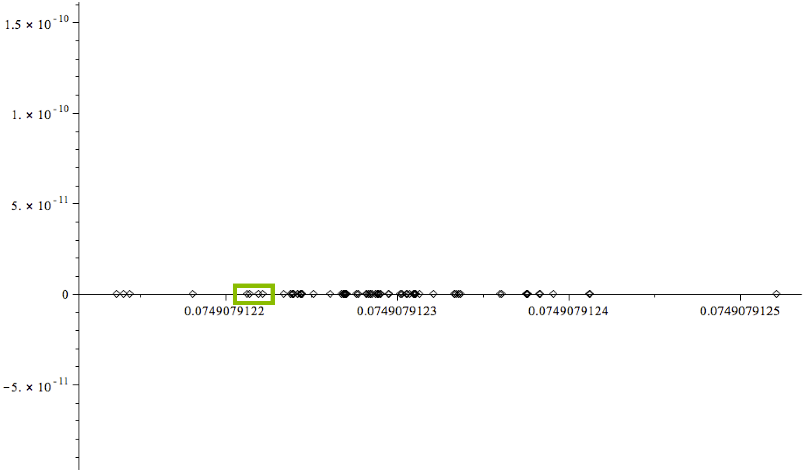

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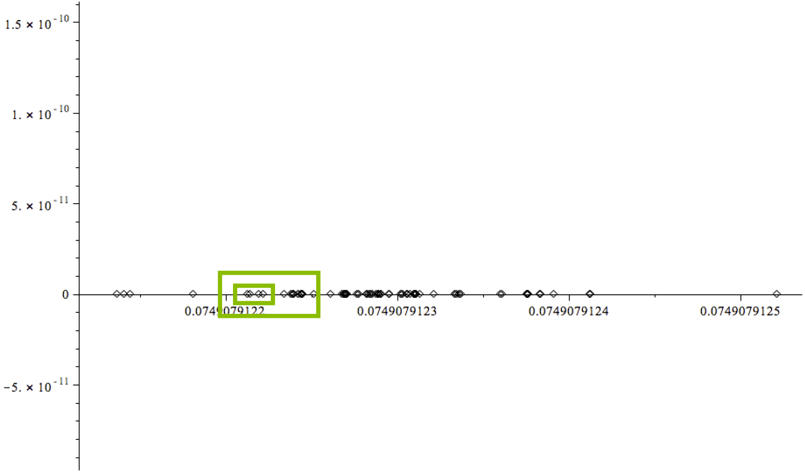

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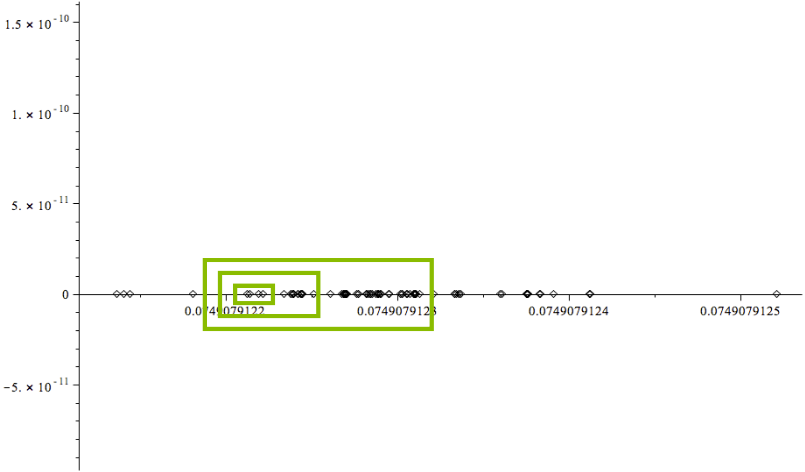

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Assuming distinct roots will give very wrong estimates for α in the next step. Instead, we *reformulate* each of the D-finite models to reflect that the singularity has multiplicity 3. Then, the α estimates are much better.

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$$\left(= -\frac{256}{9\pi^2} = -2.88202477915982994 31\right).$$

Future Work

Improved Approximation Techniques:

- Logarithmic terms
- Biased approximations
- High-order sampling
- Irregular singularities