On the Growth of Merges and Staircases of Permutation Classes

Jay Pantone

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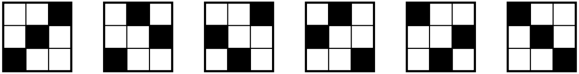
Hanover, NH



AMS Fall Central Sectional Meeting

October 29, 2016

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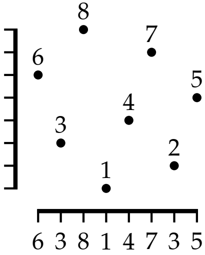
- *Example:* 63814725 $\Longleftrightarrow$



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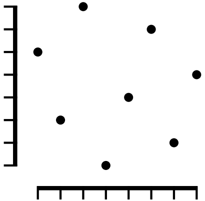

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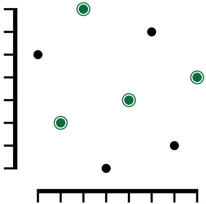

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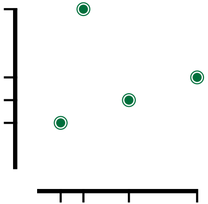

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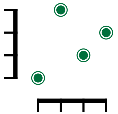
$1423 \leqslant 63814725 \quad \Longleftrightarrow$ 

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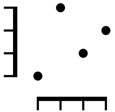

Permutation Poset

A *permutation class* is a downset in the permutation poset. In other words, if π is in the class $\mathcal{C}$ and $\sigma \leqslant \pi$, then we must have $\sigma \in \mathcal{C}$.

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The class with basis B is denoted $\mathrm{Av}(B)$.

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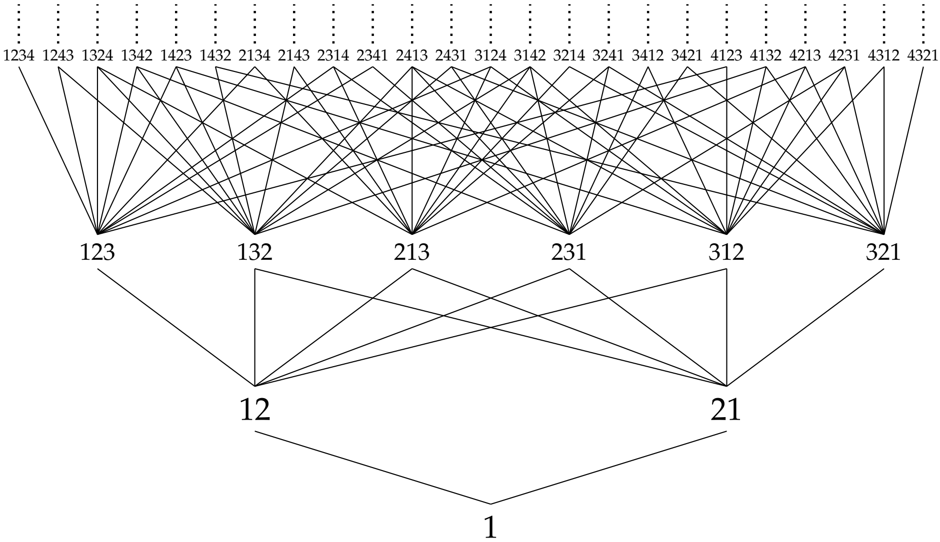

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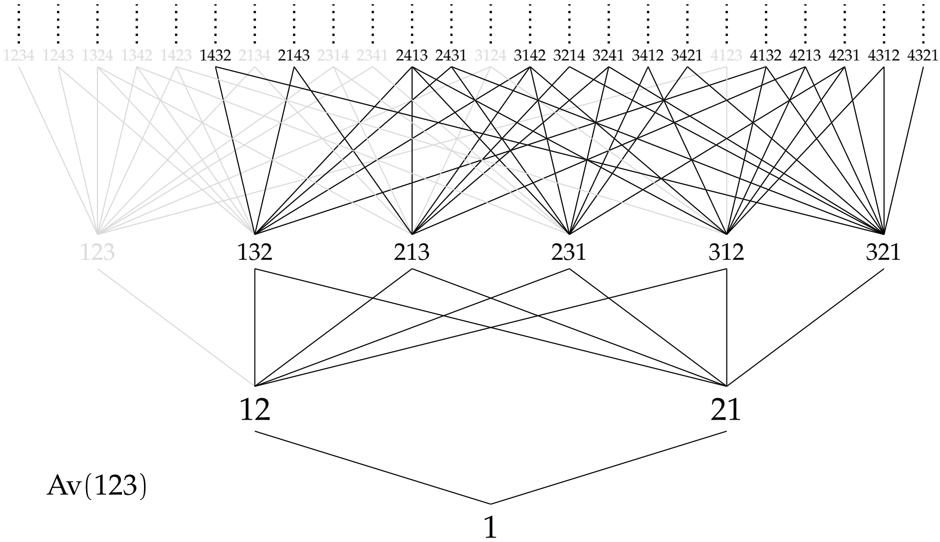

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if it exists, and most of us think it always does.

Example: The class $\mathrm{Av}(231)$ is counted by the Catalan numbers, so $\mathrm{gr}(\mathrm{Av}(231)) = 4$.

Motivation

It's been known for quite a while that

$$\mathrm{gr}(\mathrm{Av}(k \cdots 21)) = (k-1)^2.$$

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It's fairly simple to prove that $(k-1)^2$ is an upper bound, but proving that it's a lower bound is much harder.

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A permutation is in the *merge* of $\mathcal{C}$ and $\mathcal{D}$, denoted $\mathcal{C} \odot \mathcal{D}$, if there is a red-blue coloring of its entries such that the blue entries form a permutation in $\mathcal{C}$ and the red entries form a permutation in $\mathcal{D}$.

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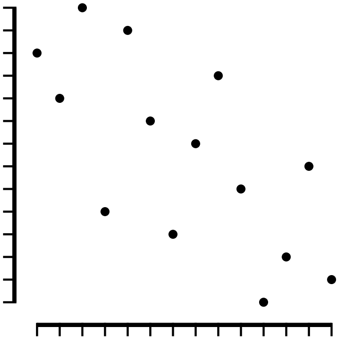

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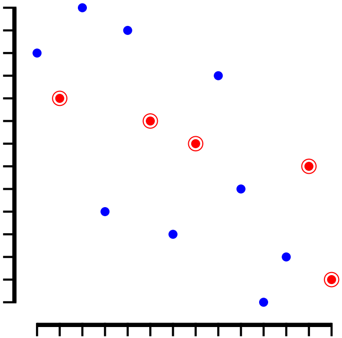

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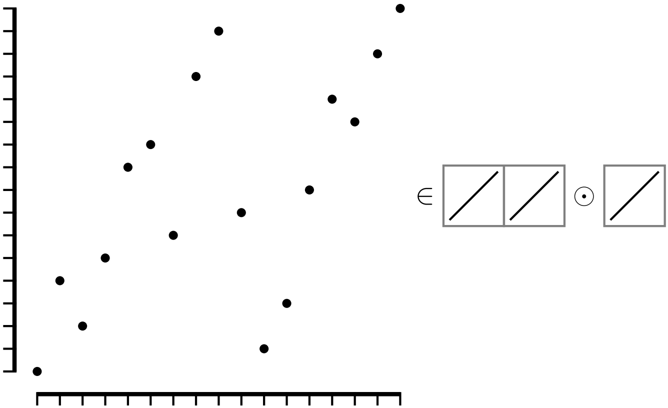

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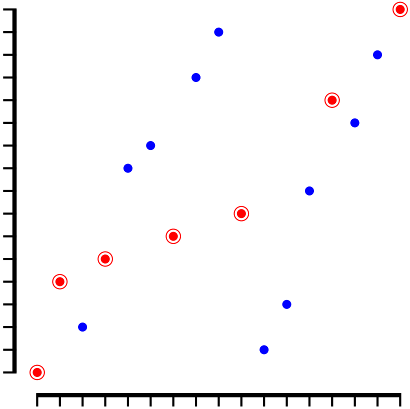
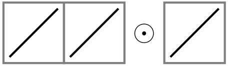
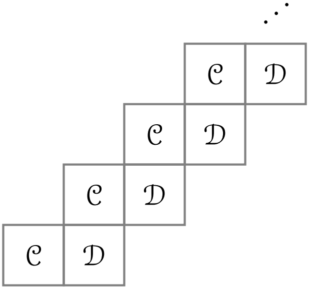


$\in$ 

Staircases

The $(\mathcal{C}, \mathcal{D})$-staircase is the set of all permutations that can be partitioned into the form



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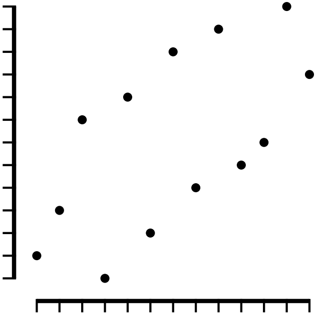
Example: The $(\text{Av}(21), \text{Av}(21))$-staircase:



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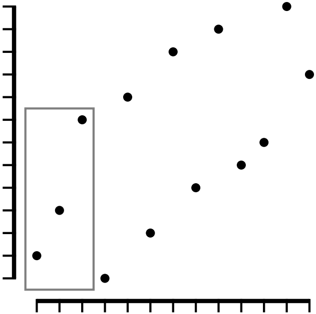

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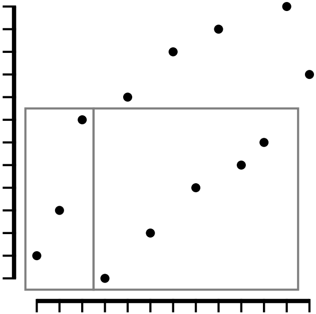

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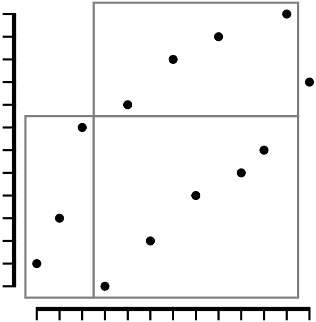

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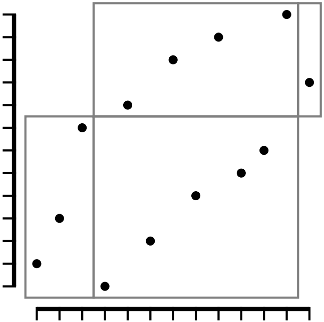

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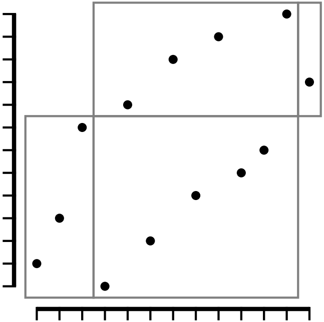

This is a particularly nice case: the $(\mathrm{Av}(21), \mathrm{Av}(21))$-staircase is exactly $\mathrm{Av}(321)$.

Bounds

Theorem. (Claesson, Jelínek, Steingrímsson) For any two classes $\mathcal{C}$ and $\mathcal{D}$,

$$\overline{\mathrm{gr}}(\mathcal{C} \odot \mathcal{D}) \leqslant \left(\sqrt{\overline{\mathrm{gr}}(\mathcal{C})} + \sqrt{\overline{\mathrm{gr}}(\mathcal{D})} \right)^2$$

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Idea:

$$|(\mathcal{C} \odot \mathcal{D})_n| \leqslant \sum_{i=0}^{n} \binom{n}{i}^2 |\mathcal{C}_i||\mathcal{D}_{n-i}|.$$

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Theorem. (Albert, P., Vatter) For any two classes $\mathcal{C}$ and $\mathcal{D}$,

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Idea: Take the "limit" of the following theorem of Albert and Vatter.

Let $\mathcal{M}$ be a $t \times u$ matrix of permutation classes, each with a proper growth rate, and define the $t \times u$ matrix Γ by $\Gamma_{k,\ell} = \sqrt{\mathrm{gr}(\mathcal{M}_{k,\ell})}$. The growth rate of $\mathrm{Grid}(\mathcal{M})$ is equal to the greatest eigenvalue of $\Gamma^T \Gamma$ (or equivalently, of $\Gamma \Gamma^T$).

Bounds

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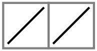
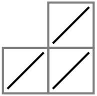


$$\longrightarrow \quad \begin{pmatrix} 1 & 1 \end{pmatrix} \quad \longrightarrow \quad \mathrm{gr}(\mathcal{C}) \geqslant 2$$

Bounds



$$\longrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \longrightarrow \mathrm{gr}(\mathcal{C}) \geqslant 1 + \phi \approx 2.618$$

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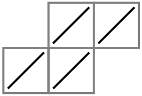
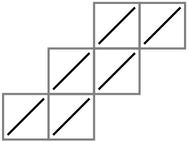


$$\longrightarrow \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \longrightarrow \mathrm{gr}(\mathcal{C}) \geqslant 3$$

Bounds



$$\longrightarrow \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \mathrm{gr}(\mathcal{C}) \approx 3.414$$

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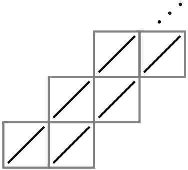

$$\longrightarrow \begin{pmatrix} 0 & 0 & 0 & \iddots \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \longrightarrow \quad \mathrm{gr}(\mathcal{C}) \to 4.$$

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Sometimes for a given class $\mathcal{E}$, we can find classes $\mathcal{C}$ and $\mathcal{D}$ such that

$$\left[(\mathcal{C}, \mathcal{D})\text{-staircase}\right] \subseteq \mathcal{E} \subseteq \mathcal{C} \odot \mathcal{D}$$

and conclude

$$\mathrm{gr}(\mathcal{E}) = \left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})} \right)^2.$$

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By the results on the previous page

$$\begin{aligned} \mathrm{gr}(k \cdots 21) &= \left(\sqrt{\mathrm{gr}(\mathrm{Av}((k-1) \cdots 21))} + \sqrt{\mathrm{gr}(\mathrm{Av}(21))} \right)^2 \\ &= \left(\sqrt{(k-2)^2} + \sqrt{1^2} \right)^2 \\ &= (k-1)^2. \end{aligned}$$

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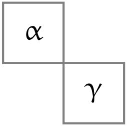
[Note: The growth rate of classes avoiding a single pattern always exist!]

Other Applications

Skew sum:

$\alpha \ominus \gamma =$ 

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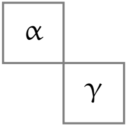
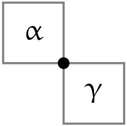
$\alpha \ominus \gamma =$ 

$\alpha \ominus 1 \ominus \gamma =$ 

Other Applications

Let α and γ be any permutations. Define

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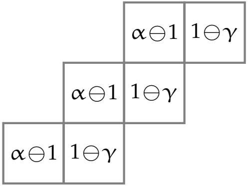

Other Applications

Theorem. (Bóna) For all permutations α and γ

$$\mathrm{gr}(\mathrm{Av}(\alpha \ominus 1 \ominus \gamma)) = \left(\sqrt{\mathrm{gr}(\mathrm{Av}(\alpha \ominus 1))} + \sqrt{\mathrm{gr}(\mathrm{Av}(1 \ominus \gamma))} \right)^2.$$

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Corollary. (Bóna)

$$\mathrm{gr}(\mathrm{Av}(54213)) = \left(1 + \sqrt{\mathrm{gr}(\mathrm{Av}(4213))}\right)^2 = 9 + 4\sqrt{2}.$$

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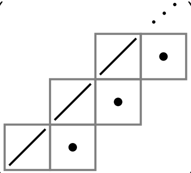
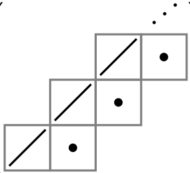
First known example of a non-integral growth rate of a principal class.

Staircase Mysteries

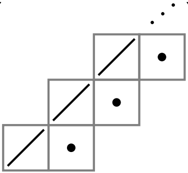
gr () $\geqslant \left(\sqrt{1} + \sqrt{0}\right)^2 = 1.$

Staircase Mysteries

gr $\left(\right.$  $\left. \right) \geqslant \left(\sqrt{1} + \sqrt{0}\right)^2 = 1.$

Actual growth rate: $1 + \varphi \approx 2.618$.

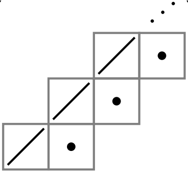
Staircase Mysteries

$\mathrm{gr}\Bigg($  $\Bigg) \geqslant \left(\sqrt{1} + \sqrt{0}\right)^2 = 1.$

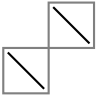
Actual growth rate: $1 + \varphi \approx 2.618$.

Growth Rate: 1.

Staircase Mysteries

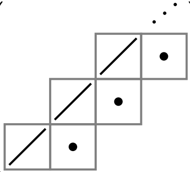
gr() $\geqslant \left(\sqrt{1}+\sqrt{0}\right)^2 = 1.$

Actual growth rate: $1+\varphi \approx 2.618$.

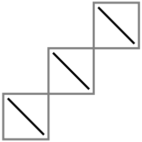


Growth Rate: 1.

Staircase Mysteries

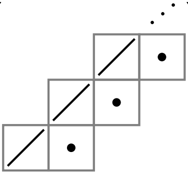
gr (

) $\geqslant \left(\sqrt{1} + \sqrt{0}\right)^2 = 1.$

Actual growth rate: $1 + \varphi \approx 2.618$.

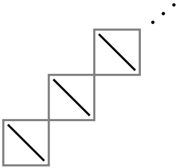


Growth Rate: 1.

Staircase Mysteries

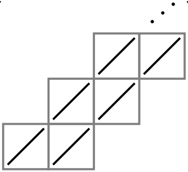
gr() $\geqslant \left(\sqrt{1} + \sqrt{0}\right)^2 = 1.$

Actual growth rate: $1 + \varphi \approx 2.618$.

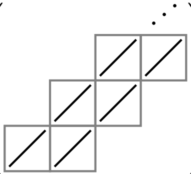


Growth Rate: **2**.

Staircase Mysteries

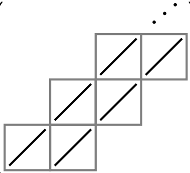
$\mathrm{gr}\Bigg($  $\Bigg) \geqslant \left(\sqrt{1}+\sqrt{1}\right)^2 = 4.$

Staircase Mysteries

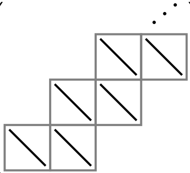
gr () $\geqslant \left(\sqrt{1}+\sqrt{1}\right)^2 = 4.$

Actual growth rate: 4.

Staircase Mysteries

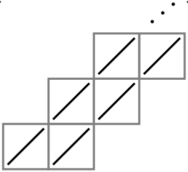
gr () $\geqslant \left(\sqrt{1} + \sqrt{1}\right)^2 = 4.$

Actual growth rate: 4.

gr () $\geqslant \left(\sqrt{1} + \sqrt{1}\right)^2 = 4.$

Staircase Mysteries

$\mathrm{gr}\Bigg($



$\Bigg) \geqslant \left(\sqrt{1} + \sqrt{1}\right)^2 = 4.$

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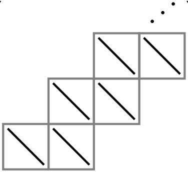
$\mathrm{gr}\Bigg($



$\Bigg) \geqslant \left(\sqrt{1} + \sqrt{1}\right)^2 = 4.$

Actual growth rate: ≈ 4.5189296. (exact value unknown)

Merge Mysteries

Theorem. (Albert, P., Vatter)

$$\mathrm{gr}(\mathrm{Av}(\alpha) \odot \mathrm{Av}(\beta)) = \left(\sqrt{\mathrm{gr}(\mathrm{Av}(\alpha))} + \sqrt{\mathrm{gr}(\mathrm{Av}(\beta))} \right)^2 .$$

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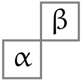
Idea:

- *sum-indecomposable* means not of the form 

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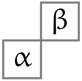
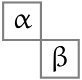
- *sum-indecomposable* means not of the form 

- *skew-indecomposable* means not of the form 

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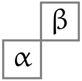
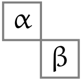
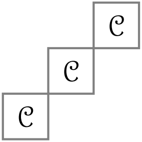
- *sum-indecomposable* means not of the form 

- *skew-indecomposable* means not of the form 

- If α is sum-indecomposable, then $\mathcal{C} = \mathrm{Av}(\alpha)$ is *sum-closed*:



Merge Mysteries

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Merge Mysteries

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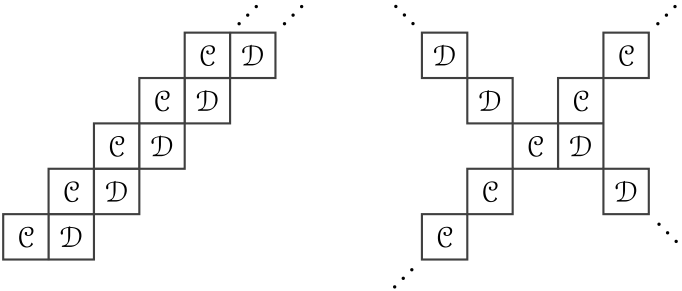

Merge Mysteries

There are *no known examples* of classes $\mathcal{C}$ and $\mathcal{D}$ such that

$$\mathrm{gr}(\mathcal{C} \odot \mathcal{D}) \neq \left(\sqrt{\mathrm{gr}(\mathcal{C})} + \sqrt{\mathrm{gr}(\mathcal{D})} \right)^2 .$$

Merge Mysteries

There are *no known examples* of classes $\mathcal{C}$ and $\mathcal{D}$ such that

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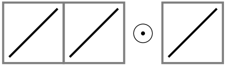
Possible Counterexample?

Does gr () equal $3 + 2\sqrt{2}$?

Thanks!