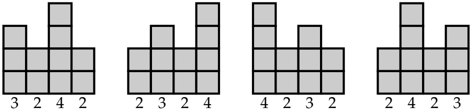
Equivalence of Words in the Generalized Factor Order

(joint work with Brian Miceli and Vince Vatter)

Jay Pantone
University of Florida



AMS Section Meeting – Washington, DC

March 7, 2015

LETTERS & WORDS

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Example: In the factor order, bca is a factor of ad<u>bca</u>d.

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Example: When $P = \mathbb{P}$, the positive integers under the normal order, we have $1423314 \geq_{\text{gfo}} 3123$ because the subword 4233 dominates 3123.

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Let $u \in \mathbb{P}^*$. Define the generating function

$$A_u(x, y, z) = \sum_{w \in \mathbb{P}^*} x^{|w|} y^{\|w\|} z^{\text{\# of factors dominating } u}.$$

AVOIDANCE IN THE GENERALIZED FACTOR ORDER

Since

$$A_u(x, y, z) = \sum_{w \in \mathbb{P}^*} x^{|w|} y^{\|w\|} z^{\#\text{ of factors dominating } u},$$

the generating function (by length and weight) of words which avoid u is $A_u(x, y, 0)$.

WILF-EQUIVALENCE

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This is a different definition of strong Wilf-equivalence given by Kitaev, Liese, Remmel, and Sagan.

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Pantone and Vatter proved that if $A_u(x, y, z) = A_v(x, y, z)$, then u and v are rearrangements, and based on numerical evidence we strongly believe that $u \sim v \implies u \sim_s v$, i.e.,

$$[A_u(x, y, 0) = A_v(x, y, 0)] \implies [A_u(x, y, z) = A_v(x, y, z)].$$

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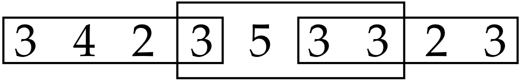

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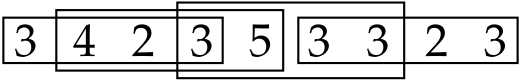

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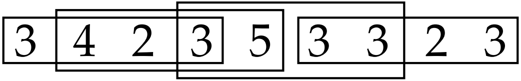

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Define the *cluster generating function* by

$$C_u(x, y, z) = \sum_{m \geq 1} z^m \sum_{\substack{m\text{-clusters} \\ c \text{ of } u}} x^{|c|} y^{\|c\|}.$$

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To find the generating function where the coefficient of $x^\ell y^w z^m$ is the number of clusters with length ℓ, weight w, and *exactly* m occurrences of u, we just use

$$C_u(x, y, z-1).$$

This is basically an application of inclusion-exclusion.

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Any word in $\mathbb{P}^*$ can be thought of as a sequence of clusters and letters which are in no cluster. An arbitrary letter has generating function

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This tells us that

$$A_u(x, y, z) = A_v(x, y, z) \iff C_u(x, y, z-1) = C_v(x, y, z-1),$$

and so we can work with the cluster generating functions to prove the rearrangement conjecture.

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We then showed that if $M_u(x,y,z) = M_v(x,y,z)$, then u and v are rearrangements.

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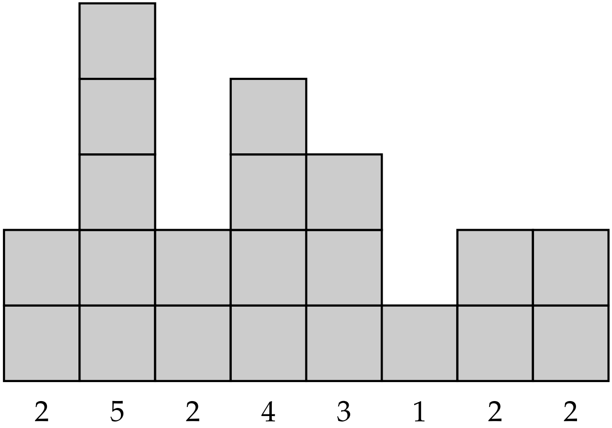

RIGID SHIFTS

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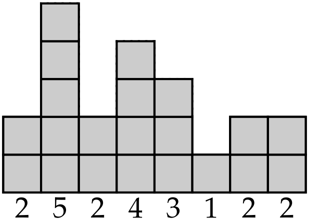

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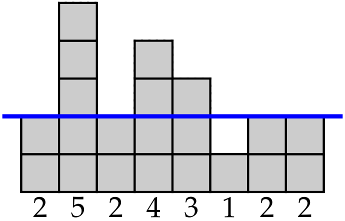

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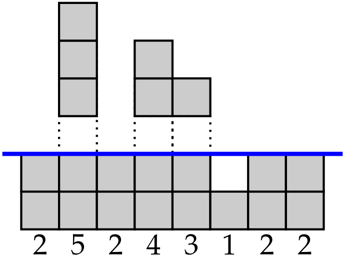

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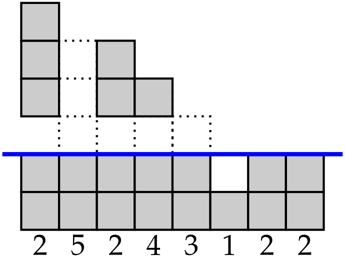

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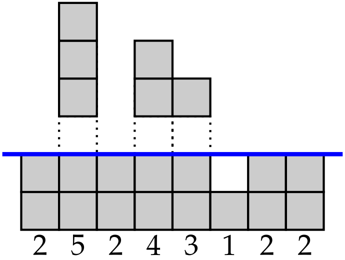

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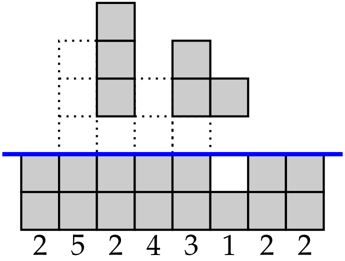

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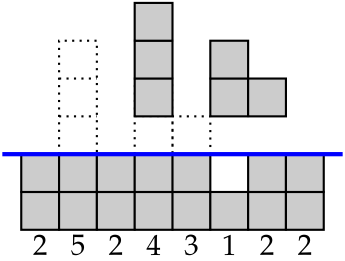

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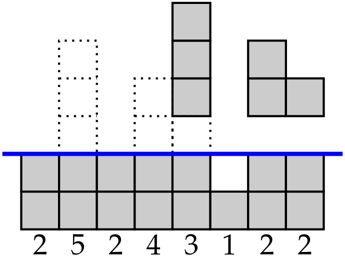

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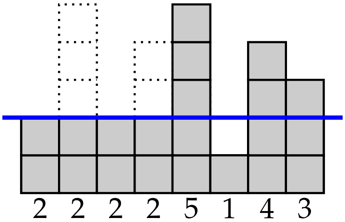

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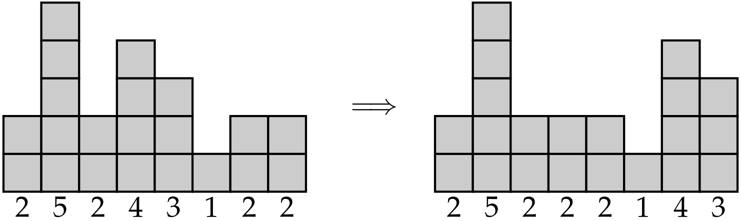
A non-example:



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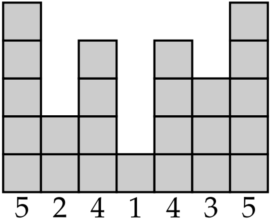
Sometimes there are no rigid shifts possible (other than reversing).



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We conjecture: **If $u \sim_s v$ then $u \sim_R v$.**

This would prove even more (especially when combined with the Rearrangement Conjecture and the $\mathsf{WE} = \mathsf{SWE}$ Conjecture).

STRONG WILF-EQUIVALENCE OF RIGID SHIFTS

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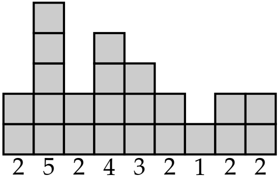

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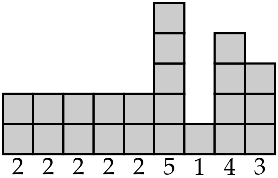
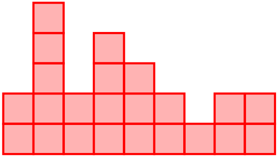


Strong Wilf-Equivalence of Rigid Shifts



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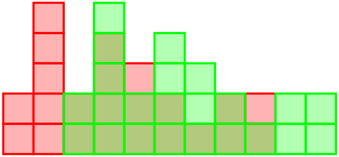

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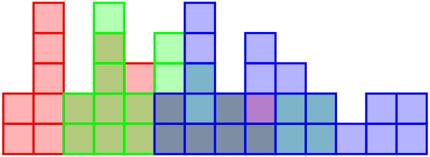

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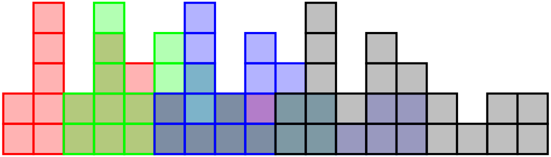

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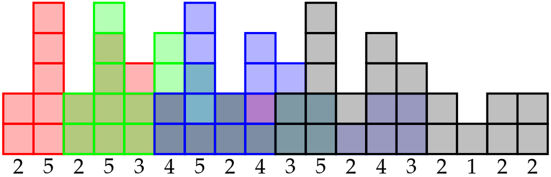

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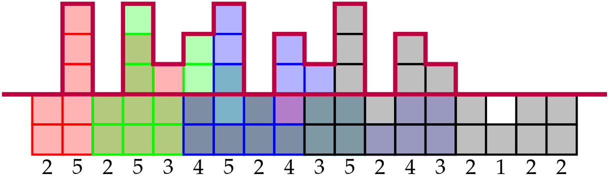

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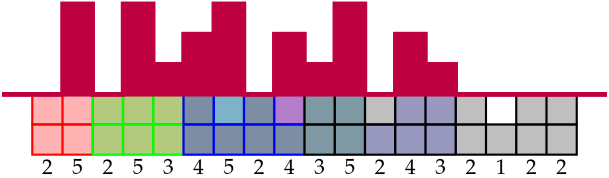

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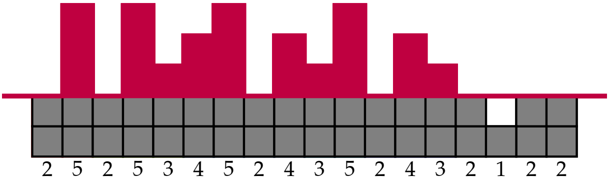

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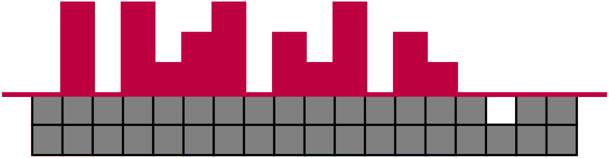

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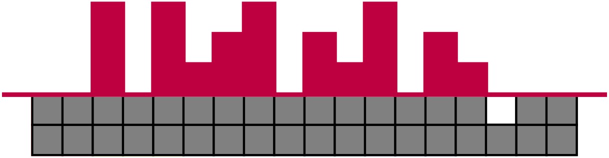

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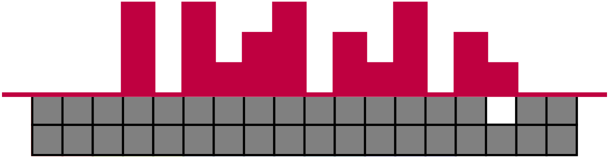

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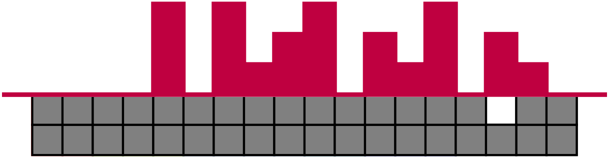

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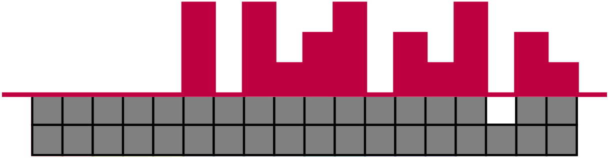

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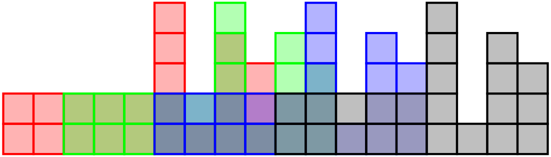

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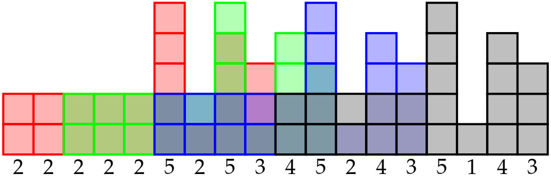

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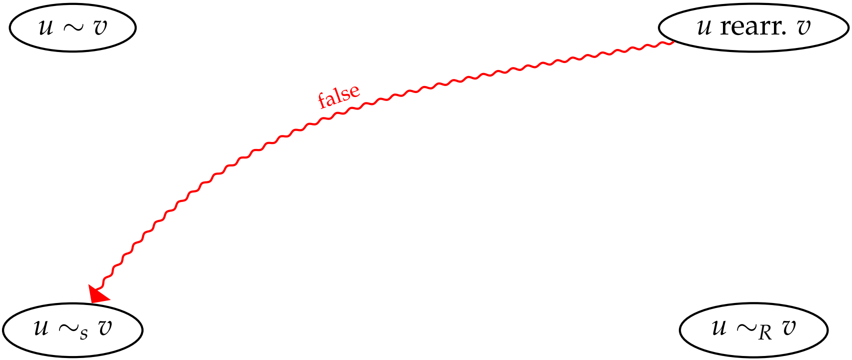

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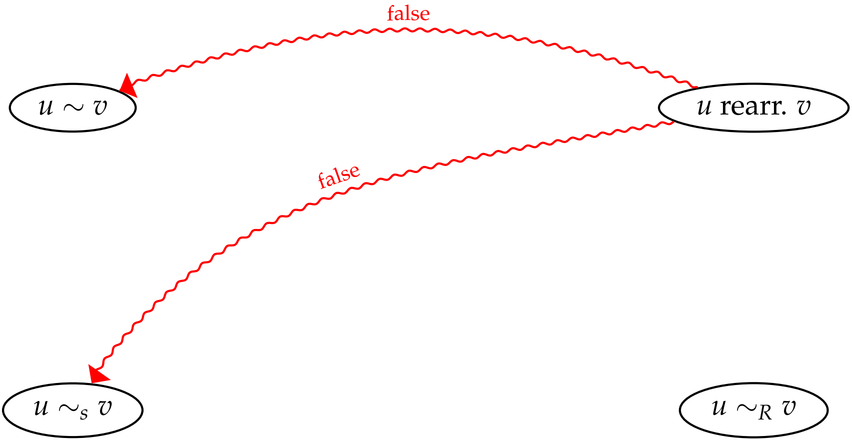

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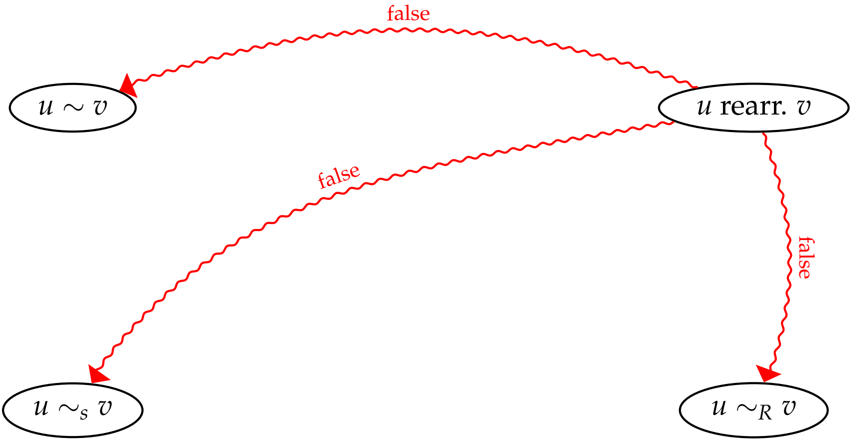

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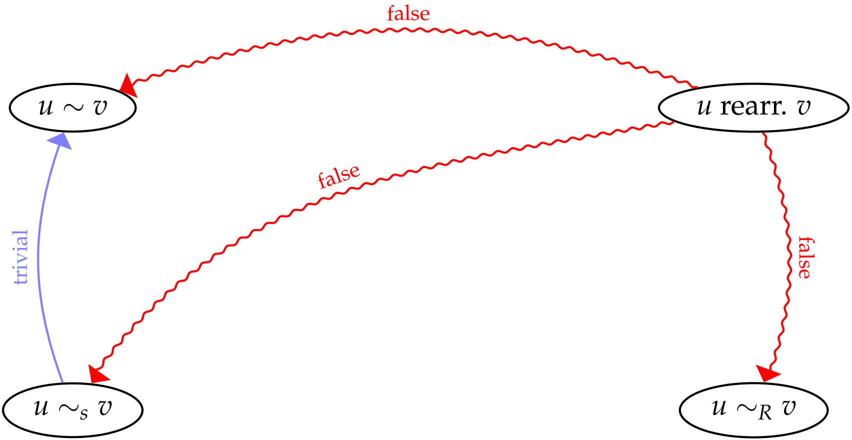

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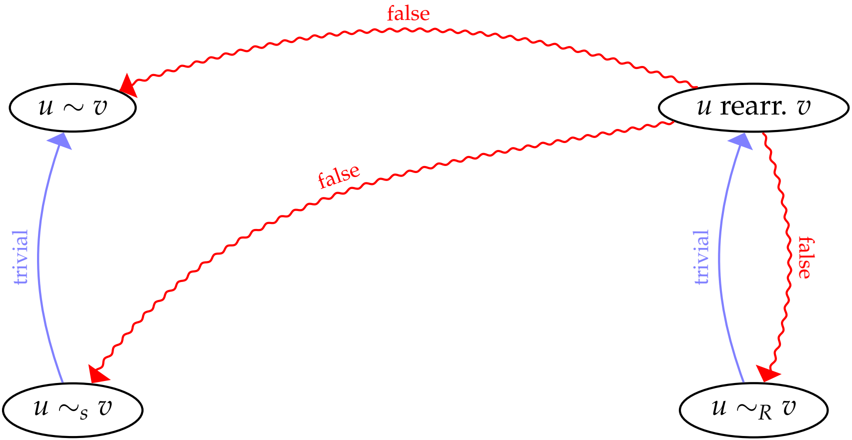

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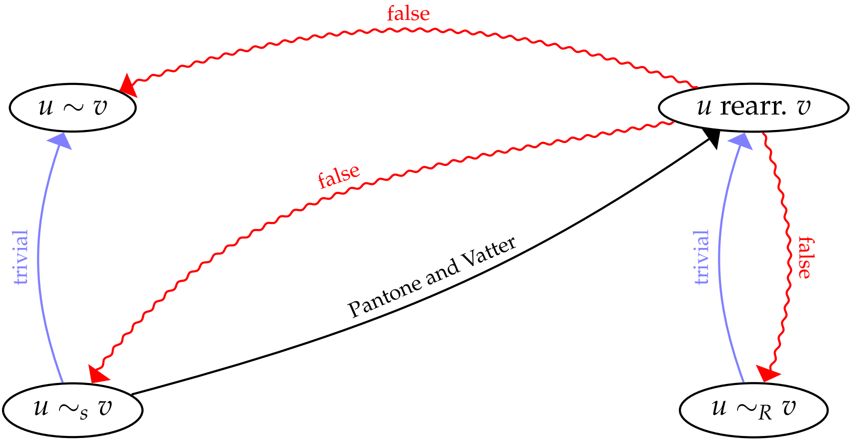

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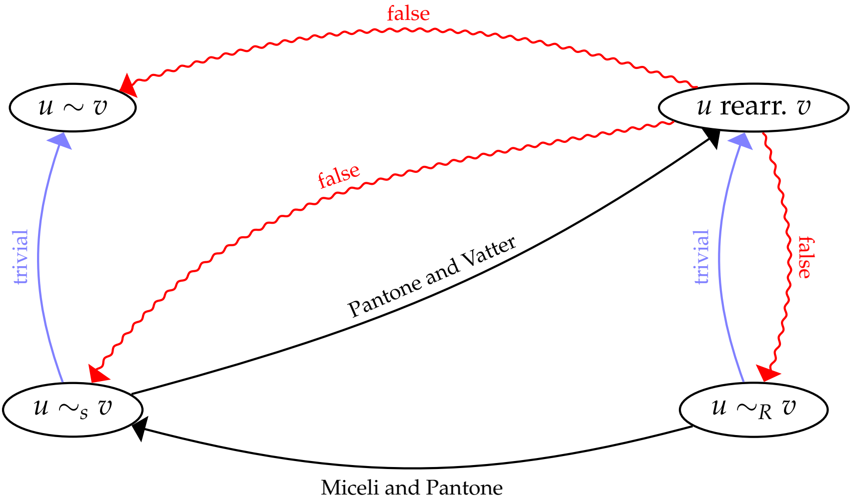

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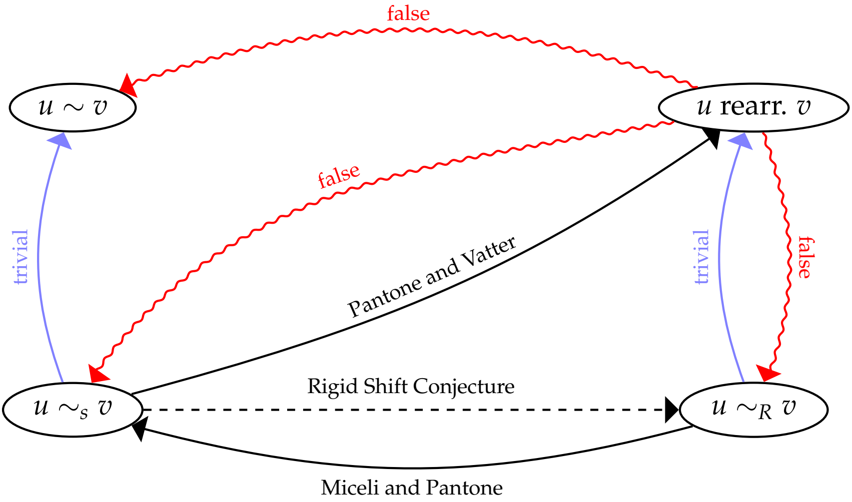

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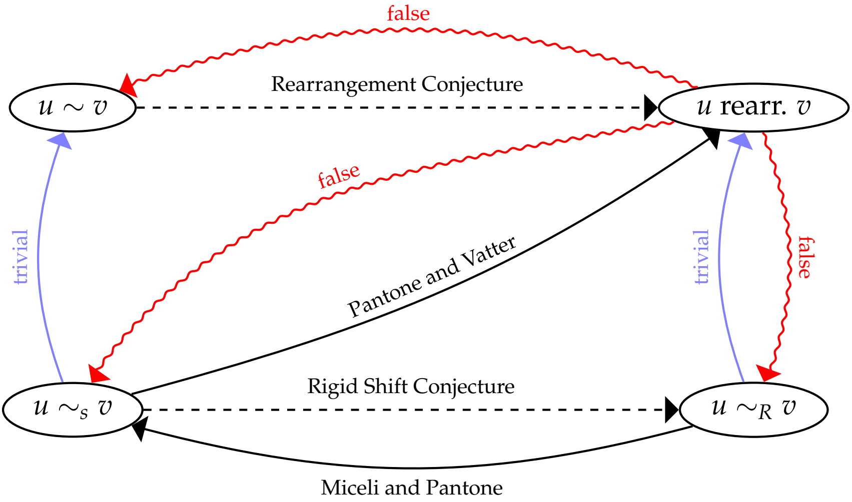

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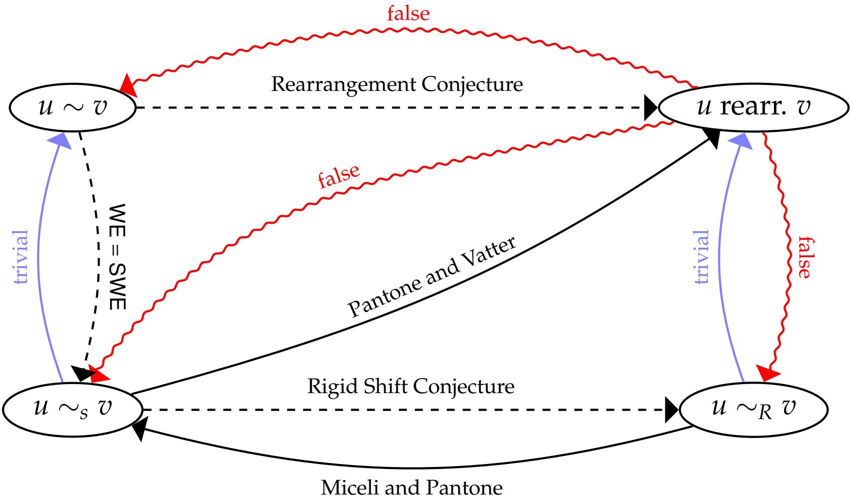

Thanks for coming! Any questions?