Math 28 – Notes on Functions

Question 7 asks you to count the number of functions with certain properties. The way we're talking about functions feels much different than what you're used to in calculus because for now we're only working with functions whose domains and codomains are *finite sets*. I'll explain a little bit here, and give you some hints to get started on #7. You should also read Section 1.2.2 on Functions and Digraphs.

Let S and T be finite sets. When we want to say that f is a function from S to T, we write

$$f : S \to T.$$

This doesn't tell you what the function does! It just tells you what the domain and codomain are.

The function itself is just a rule that says "When you get a certain input, this is your output."

Here's an example: Let S be the set of states in the US (so $|S| = 50$), and let $T = \{\texttt{yes}, \texttt{no}\}$ (so $|T| = 2$). Let $s \in S$ be a state and define

$$v(s) = \begin{cases} \texttt{yes}, & \text{if the name of the state has at least four vowels} \\ \texttt{no}, & \text{if the name of the state does not have at least four vowels} \end{cases}.$$

(Let's say that `y` is not a vowel for our purposes.)

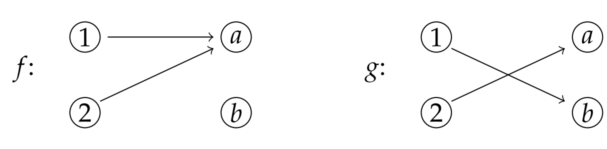
With this definition, $v(\texttt{Florida}) = \texttt{no}$ and $v(\texttt{New Hampshire}) = \texttt{yes}$. If you want to draw this function as a digraph (see section 1.2.2), you'd have 50 circles on the left, two on the right, and every circle on the left would have an arrow pointing from it to exactly one of the arrows on the right.

Question 7a asks you to enumerate ("enumerate" = "count") the number of functions $f : \{1, 2\} \to \{a, b\}$. If you're confused, think about drawing digraphs again: you have two circles on the left (for 1 and 2), two circles on the right (for a and b), and you need to count the number of ways to add arrows from 1 and 2 to a and b. Here are two of the possibilities, and I'll let you find the rest:

$$f(1) = a, f(2) = a$$

$$g(1) = b, g(2) = a$$

These correspond to the digraphs:



(*Hint:* Remember our discussion on *choices* from Friday.)

There are a few properties that functions can have. Some are mentioned in Question 7, some will come later.

- A function $f : S \to T$ is *one-to-one* or *injective* if no two elements in S point to the same element of T. The function f above is not injective, because 1 and 2 point to the same element of T. The function g is injective.
- A function $f : S \to T$ is *onto* or *surjective* if every element in T is pointed to by at least one element of S. The function f above is not surjective because nothing points to b. The function g is surjective.
- A function is *bijective* if it's both injective and surjective. Bijections are very important in combinatorics, and we'll talk more about them on Monday. f is not a bijection, but g is.

The state vowel function v is surjective (because there are states with at least four vowels and states with less than four vowels), but it's not injective. If we defined a similar function $\widehat{v}$ that equals `yes` when a state name has at least fifty vowels, and `no` when it has less than fifty vowels, then obviously $\widehat{v}$ is not surjective.