MATH 28 – HOMEWORK 9

due Wednesday, March 8

There will be no resubmissions of Homework 9 allowed.

In this homework assignment, we will prove Newton's Generalized Binomial Theorem using several steps.

Theorem: For any real number α and positive integer k, define

$$\binom{\alpha}{k} = \frac{\alpha^{\underline{k}}}{k!},$$

and define $\binom{\alpha}{0} = 1$. Then

$$(1+x)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} x^k.$$

(Because this is not an analysis class, we're not worried about the convergence of this Taylor series.)

Define

$$f(x) = (1+x)^\alpha \qquad g(x) = \sum_{k \geq 0} \binom{\alpha}{k} x^k.$$

Our goal is to show that $f(x) = g(x)$.

1. Show that $f(x)$ satisfies the differential equation

   $$(1+x)y'(x) = \alpha y(x).$$

   (In other words, show that when you plug in $y(x) = f(x)$, the equation is true.)

2. Show that $g(x)$ also satisfies the differential equation

   $$(1+x)y'(x) = \alpha y(x).$$

3. The fact that two functions are both solutions of a differential equation is not enough to show that they're equal. First, let $y(x)$ any solution of the differential equation, and define $H(x) = \dfrac{y(x)}{f(x)}$. Show that $H'(x) \equiv 0$ (i.e., show that $H'(x) = 0$ for all values of x). Explain why this implies that if two functions satisfy the differential equation, then one must be a multiple of the other.

4. Combine the previous three questions to show that $f(x) = g(x)$. (*Hint:* Find $f(0)$ and $g(0)$.)