Math 28 – Homework 9

due Wednesday, March 8

There will be no resubmissions of Homework 9 allowed.

In this homework assignment, we will prove Newton’s Generalized Binomial Theorem using several steps.

Theorem: For any real number \( \alpha \) and positive integer \( k \), define

\[
\binom{\alpha}{k} = \frac{\alpha^k}{k!},
\]

and define \( \binom{\alpha}{0} = 1 \). Then

\[
(1 + x)^\alpha = \sum_{k \geq 0} \binom{\alpha}{k} x^k.
\]

(Because this is not an analysis class, we’re not worried about the convergence of this Taylor series.)

Define

\[
f(x) = (1 + x)^\alpha \quad \text{and} \quad g(x) = \sum_{k \geq 0} \binom{\alpha}{k} x^k.
\]

Our goal is to show that \( f(x) = g(x) \).

1. Show that \( f(x) \) satisfies the differential equation

\[
(1 + x)y'(x) = ay(x).
\]

(In other words, show that when you plug in \( y(x) = f(x) \), the equation is true.)

2. Show that \( g(x) \) also satisfies the differential equation

\[
(1 + x)y'(x) = ay(x).
\]

3. The fact that two functions are both solutions of a differential equation is not enough to show that they’re equal. First, let \( y(x) \) any solution of the differential equation, and define \( H(x) = \frac{y(x)}{f(x)} \). Show that \( H'(x) \equiv 0 \) (i.e., show that \( H'(x) = 0 \) for all values of \( x \)). Explain why this implies that if two functions satisfy the differential equation, then one must be a multiple of the other.

4. Combine the previous three questions to show that \( f(x) = g(x) \). (Hint: Find \( f(0) \) and \( g(0) \).)