MATH 28 – HOMEWORK 3

due Wednesday, January 25

1. (#57) Explain in at least two different ways why

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k}.$$

   Find one combinatorial proof (think of the examples from class) and find a completely different proof that uses the Binomial Theorem.

2. (#59) Use the Binomial Theorem to calculate a nicer expression for

$$\sum_{i=0}^{n} i\binom{n}{i}.$$

   Hint: Think about taking a derivative.

3. (#64) All the powers of five end in a five, and all the powers of two are even. Show that for some integer n, if you take the first n powers of a prime other than two or five, one must have "01" as the last two digits.

   Hint: Use the pigeonhole principle.

   Example: The smallest n such that 7^n ends in the digits is "01" is 4 ($7^4 = 2401$). The smallest n such that 19^n ends in the digits "01" is 10 ($19^{10} = 6131066257801$). Your proof won't actually calculate the appropriate n for you, it will just show that such an n always exists.

   Another hint: The theorem is clearly false for the primes 2 and 5, therefore your theorem *must* depend on the fact that your prime is not 2 or 5! If your theorem "works" for 2 and 5, then something must be wrong with it.

4. (Kind of #66) In this question we'll prove that the Ramsey number $R(3,3)$ must be at least 6. We'll show this by proving that for $n = 3,4,5$ it's possible to have a group of n people for which no subset of three people all know each other and for which no subset of three people all don't know each other. (Assume "knowing" is symmetric: if A knows B then B knows A.) Equivalently, we'll show that it's possible to color the edges of the graphs K_3, K_4 and K_5 with two colors such that there is no monochromatic K_3 subgraph.

   a) Draw three vertices, labeled "Alice", "Brian", and "Cy". The complete graph on these three vertices has three edges. Draw these three edges, each in red or blue, in such a way so there is no monochromatic K_3 subgraph. (Don't overthink this one...)

   b) Draw four vertices, labeled "Alice", "Brian", "Cy", and "Dominic". The complete graph on these four vertices has six edges. Draw these six edges, each in red or blue, in such a way so there is no monochromatic K_3 subgraph.

c) Draw five vertices, labeled "Alice", "Brian", "Cy", "Dominic", and "Greta". The complete graph on these five vertices has ten edges. Draw these ten edges, each in red or blue, in such a way so there is no monochromatic K_3 subgraph.

5. (Kind of #65)

a) Show that in a set of six people, there is a set of at least three people who all know each other, or a set of at least three people none of whom know each other. (Assume "knowing" is symmetric: if A knows B then B knows A.)

b) Explain why the fact in the previous part, together with the previous problem, shows that $R(3,3) = 6$.

6. (Chapter 1 Supplementary Questions #1 and #2)

a) We can write the number n as a sum of n ones (which uses $n-1$ plus signs in between the ones). In how many ways may we write n as an ordered sum of k positive numbers? These objects are called *compositions of n into k parts*. (Here, "ordered" means that $1+1+2$, $1+2+1$, and $2+1+1$ are all different compositions of 4 into 3 parts.) Note that the parts cannot be zero, they must be at least 1.

Hint: Think about the explanation for the "stars and bars" technique that used the robot analogy.

b) What is the total number of compositions of n (into any number of parts)?

Hint: What is the number of subsets of a set of size $n-1$? How does that apply here?