MATH 28 – ADDITIONAL QUESTIONS

These are additional questions for practice, homework, and exam review. Some are just reworded versions of questions from the textbook, but most are brand new.

1. PIGEONHOLE PRINCIPLE

E1. A busy airport sees 1500 takeoffs per day. Prove that there are two planes that must take off within a minute of each other.

E2. Prove that when ten points are picked randomly inside a square of side length 1, there will always be at least two points within a distance of 0.48 from each other.

Hint 1: Divide the square into 9 parts.

Hint 2: $\frac{\sqrt{2}}{3} \approx 0.4714$

E3. Prove that when ten points are picked randomly inside a square of size length 1, there will always be at least three points that can be covered by a circle of radius $1/2$.

E4. Prove that the sequence $\{1989, 19891989, 198919891989, \ldots\}$ has an element that is evenly divisible by 1991.

2. GENERATING FUNCTIONS

E5. Find the generating function of each sequence:

(a) $a_n = n^2$

(b) $a_n = n^3$

(c) $a_n = An^3 + Bn^2 + Cn + D$

(d) $a_n = 3^n$

(e) $a_n = n^2 3^n$

(f) $a_n = n^2 3^{n-1}$

E6. Find the generating function for the sequence that satisfies the recurrence:

(a) $a_n = 3a_{n-1} + 2$, $a_0 = 5$

(b) $a_n = 2a_{n-1} + 3a_{n-2}$, $a_0 = 2$, $a_1 = 1$

(c) $a_n = na_{n-1} + 2na_{n-2} + 3^n$, $a_0 = 0$, $a_1 = 1$
For this last one, you will get an equation involving $f(x)$ and $f'(x)$. Just leave your answer in this form (a differential equation) and don't try to solve it!

E7. Find a closed-form formula for the coefficients of the generating function:

(a) $A(x) = \dfrac{1 - 2x + 2x^2}{(1-x)^2(1-2x)}$

(b) $B(x) = \dfrac{1}{(1-x^2)(1-x)}$ (don't forget to factor!)

(c) $C(x) = \dfrac{1}{(1-x)(1-2x)(1-3x)}$

3. NEWTON'S GENERALIZED BINOMIAL THEOREM

E8. Recall that a multiset is a collection of elements for which the order does not matter, but repeats are allowed. For example $\{1, 1, 1, 3, 4, 4\}$ is a multiset with six elements.

Given a **set** S, a **multiset subset** of S is a subset of S in which repetition is allowed. For example, if $S = \{1, 2, 3, 4\}$, then one multiset subset of S is $\{1, 1, 1, 3, 4, 4\}$.

Find a formula for the number of multiset subsets of size k of an n element set.

E9. Fix n and let S be a set with n elements. Find a generating function $f(x)$ in which the coefficient of x^k is the number of multiset subsets of S of size k.

E10. Combine your answers to the two previous problems to fill in the blank:

$$(1-x)^{-n} = \sum_{k \geq 0} \underline{\hspace{3cm}} \; x^k$$

E11. Your answer to the previous problem is very helpful when finding closed-form formulas for the coefficients of rational generating functions (as we did last week). Use it to find the coefficients of the generating function

$$\frac{1}{(1-x)^5}$$

without using partial fraction decomposition.

E12. (#195) Find a formula for the power series expansion of $(1+x)^{-n}$ by filling in the blank below:

$$(1+x)^{-n} = \sum_{k \geq 0} \underline{\hspace{3cm}} \; (\underline{\hspace{1cm}})^k.$$

What does this formula say about how we should define the quantity $\binom{-n}{k}$ when n is positive?

E13. (#196) If you define $\binom{-n}{k}$ as above, then you can write down a version of the binomial theorem for $(x+y)^n$ that is valid for both nonnegative and negative integers n. Find this formula. Write down a special case with n negative, like $(x+y)^{-3}$, to see an interesting surprise that suggests why we do not use this formula later on.

(*Hint:* It may help you to write $(x+y)$ as $y\left(1+\frac{x}{y}\right)$.)

E14. For any real number α, define

$$\binom{\alpha}{k} = \frac{\alpha^{\underline{k}}}{k!}.$$

Verify that if n is a (negative or nonnegative) integer and k is a nonnegative integer than this formula is equivalent to the one found above.

Compute $\binom{1/2}{5}$.

E15. **Newton's Generalized Binomial Theorem:** For all real numbers α:

$$(1+x)^{\alpha} = \sum_{k \geq 0} \binom{\alpha}{k} x^k.$$

Use this theorem to find a closed-form formula for the coefficients of the generating function

$$\sqrt{1-4x}.$$

E16. The function

$$\frac{1}{\sqrt{1-4x}}$$

is the generating function for what sequence? What combinatorial problem have we seen for which this is the counting sequence?