NAME : _____

Math 28

Midterm 2
February 22, 2017

Prof. Pantone

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have 120 minutes and you should attempt all problems.

- Print your name in the space provided.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

TIPS:

- You don't have numerically expand all answers. For example, you can leave an answer in the form $10! \cdot \binom{5}{3}^2$, rather than 362880000.
- Use scratch paper to figure out your answers and proofs before writing them on your exam.
- Work cleanly and neatly; this makes it easier to give partial credit.

Problem	Points	Score
1	40	
2	15	
3	15	
4	15	
5	15	
6	0	
7	0	
Total	100	

Section 1: Short Answer Questions.

1. (40) Each question in this section is worth 8 points. You should **justify your answers**, but the justification should be concise (a few sentences at most).

(a) Let a_n denote the number of ways to pay n dollars using one-dollar bills, five-dollar bills, and ten-dollar bills. Find the generating function for the sequence $\{a_n\}_{n\geq 0}$.

(b) Let $S(k, n)$ denote the number of set partitions of a k element set into n parts. Find a nice formula (with no summations) for $S(k, 2)$.

(c) A senior has 37 days to write her senior thesis and she knows that it will take exactly 60 hours of work. Each day she works for a *positive integer* number of hours. How many different possible schedules are there?

(d) Let p_n denote the number of integer partitions of n into any number of parts. Find a formula in terms of the sequence $\{p_n\}_{n\geq 0}$ for the number of integer partitions of n into parts of size at most $n-4$.

(e) How many compositions of n are there into k parts, all of which are odd?

Section 3: Long Answer.

Justify all parts of your answers.

2. (15) Let p_n denote the number of integer partitions of n into any number of parts. Find a formula in terms of the sequence $\{p_n\}_{n\geq 0}$ for the number of integer partitions of n into parts that are all even.

3. (15) Consider the sequence $\{a_n\}_{n\geq 0}$ defined by the recurrence $a_n = 3a_{n-1} + 2^n$ with initial condition $a_0 = 1$. Find the generating function $f(x)$ for this sequence.

4. (15) Recall from Homework 6 that the nth Bell number is defined to be the number of all set partitions of $\{1, 2, \ldots, n\}$ with no restrictions on the number of parts.

For example, $B(3) = 5$ because the set partitions of $\{1, 2, 3\}$ are

$$\{1\}, \{2\}, \{3\} \qquad \{1, 2\}, \{3\} \qquad \{1, 3\}, \{2\}$$
$$\{2, 3\}, \{1\} \qquad \{1, 2, 3\}.$$

Also, $B(4) = 15$ and $B(5) = 52$.
Prove that $B(n) \geq 2^n$ for all $n \geq 5$.

5. (15) In how many ways can k *distinct* books be stacked in 5 *identical* boxes, if at least 3 of the boxes must be nonempty? (To clarify: the order of the books in each box matters, but the order of the boxes does not matter.)

6. (0) **BONUS: 5 points. Do not attempt the bonus problems until you have finished the rest of the exam.**

Let p_n denote the number of integer partitions of n into any number of parts. Find a formula in terms of the sequence $\{p_n\}_{n\geq 0}$ for the number of integer partitions of $2n$ into parts of size at most n. Your answer may contain a summation.

7. (0) **BONUS: 5 points. Do not attempt the bonus problems until you have finished the rest of the exam.**

Find the generating function for the number of ways to paint n ping pong balls blue and orange such that the number of blue ping pong balls is a multiple of 2 or 3 (or both) and the number of orange ping pong balls is a multiple of 4. (For the purposes of this question, assume that 0 is a multiple of 2, 3, 4.)