MATH 28 – EXAM 1 PRACTICE

Disclaimer: These practice problems are not to be considered comprehensive in any way. They are just some additional problems relating to the material we covered in class. You should be sure to study many other problems, including group work, homework, and the additional questions posted on the website.

Note: In every question from here on out, you need to prove, not just state, your answer.

1. From the symmetry of the binomial coefficients, it is not too hard to see that when n is an odd numbers, the number of subsets of $\{1, 2, \ldots, n\}$ of odd size equals the number of subsets of $\{1, 2, \ldots, n\}$ of even size. Is it true that when n is even, the same is true? Why or why not?

2. Show that among an odd number of people there is at least one person who is an acquaintance of an even number of people and therefore also a stranger to an even number of people.

3. A list of parentheses is said to be balanced if there are the same number of left parentheses as right, and as we count from left to right we always find at least as many left parentheses as right parentheses. For example $((((()()))())$ is balanced, while $((()$ and $(()()))(()$ are not. Find a bijection between balanced lists of n pairs of parentheses and diagonal lattice paths with $2n$ steps that go from $(0, 0)$ to $(2n, 0)$ and stay above the x-axis. Prove that your answer is indeed a bijection. What can you now say about the number of balanced lists of n pairs of parentheses?

4. In as many ways as you can, show that

$$\binom{n}{k}\binom{n-k}{m} = \binom{n}{m}\binom{n-m}{k}.$$

5. A tennis club has $4n$ members. To specify a doubles match, we choose two teams of two people. In how many ways may we arrange members into doubles matches so that each player is in one doubles match? In how many ways may we do it if we specify in addition who serves first on each team? (To clarify, if teams (A, B) and (C, D) are playing a doubles match together, pick which of A or B serves first on their turn and which of C or D serves first on their turn. Don't pick which team serves first in each match.)

6. A town has n streetlights running along the north side of Main Street. The poles on which they are mounted need to be painted so that they do not rust. In how many ways may they be painted with red, white, blue, and green if an even number of them are to be painted green?

7. We have n identical ping-pong balls. In how many ways may we paint them red, white, blue, and green?

8. We have n identical ping-pong balls. In how many ways may we paint them red, white, blue, and green if we use green paint on an even number of them?

9. What is the minimum number of vertices of degree one in a finite tree?

10. In a tree on any number of vertices, given two vertices how many paths can you find between them?

11. Do questions 1(a)-(d), 2(a)-(c) and 3(a)-(b) on the "Worksheet: Induction Proofs" that I passed out in class.

12. How many lattice paths are there from $(0,0)$ to $(10,10)$ that pass through $(4,4)$? How many from $(0,0)$ to (m,n) that pass through $(4,4)$? (Assume $m,n \geq 4$.)

13. Prove that $3^n > n^4$ for $n \geq 8$.

14. Prove that if n is a positive integer then $8^n - 14n + 27$ is divisible by 7.

15. Prove the identity in as many ways as you can (at least one combinatorial proof and one induction proof):
$$n(n-1)2^{n-2} = \sum_{k=2}^{n} k(k-1)\binom{n}{k}.$$