NAME : _____

Math 28

Midterm 1
February 1, 2017

Prof. Pantone

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have 120 minutes and you should attempt all problems.

- Print your name in the space provided.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

TIPS:

- You don't have numerically expand all answers. For example, you can leave an answer in the form $10! \cdot \binom{5}{3}^2$, rather than 362880000.
- Use scratch paper to figure out your answers and proofs before writing them on your exam.
- Work cleanly and neatly; this makes it easier to give partial credit.

Problem	Points	Score
1	40	
2	15	
3	15	
4	15	
5	15	
6	0	
7	0	
Total	100	

Section 1: Short Answer Questions.

1. (40) Each question in this section is worth 8 points. You should **justify your answers**, but the justification should be concise (a few sentences at most).

(a) Give the definition of the Ramsey number $R(m, n)$.

(b) How many lattice paths with steps $(1, 0)$ and $(0, 1)$ are there that start at $(0, 0)$ and go to $(2n, 2n)$ while passing through the point (n, n)?

(c) The undergraduate enrollment at Dartmouth this year is 4300. Suppose that every student is given a penny and flips it N times, recording the ordered sequence of heads and tails for each flip. What is the *maximum* possible value of N for which you can be sure that there will be at least two students that end up with identical heads/tails sequences? (Calculation help: $2^{10} = 1024$.)

(d) What is the smallest number of vertices of degree one that a tree with 50 vertices can have?

(e) The complete graph on n vertices, denoted K_n, is the graph with n vertices and all possible edges between them. Let a_n be the number of edges in K_n. Find a recurrence for a_n. Don't forget to specify initial conditions.

Section 3: Long Answer.

Justify all parts of your answers.

2. (15) Prove that $\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}$ for all n, k such that $2 \leq k \leq n-2$.

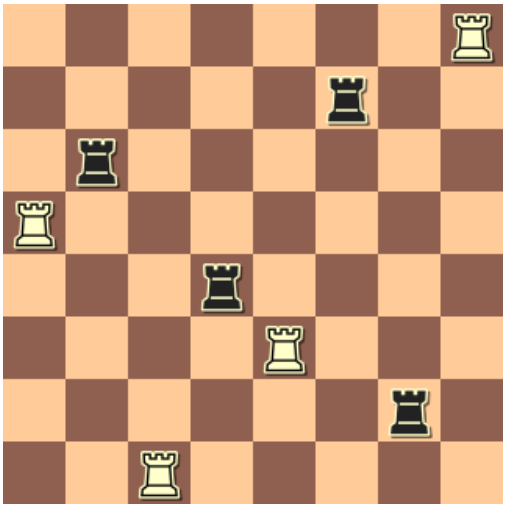
3. (15) How many subsets of the set $\{1, 2, \ldots, n\}$ are there in which either every element is even or every element is odd? (*Example.* For $n = 10$: $\{2, 8\}$, $\{1, 5, 7\}$ and $\{\}$ count, while $\{1, 2\}$ and $\{5, 6, 7\}$ don't.)

4. (15) Use induction to prove that for all positive integers n,

$$1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2.$$

If you'd like, you may use the fact that $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$.

5. (15) Consider a typical 8×8 chess board. In chess, the Rook or Castle is a piece that can move vertically or horizontally any number of spaces. How many ways are there to place four identical white rooks and four identical black rooks on a chessboard, so that no two rooks are in the same row or column. The example below shows one such configuration. Note: rotations of the board are considered different configurations.



6. (0) **BONUS: 5 points. Do not attempt the bonus problems until you have finished the rest of the exam.** Let a_n be the number of ways to climb a staircase with n steps in which you can take the steps either one at a time or two at a time. (Equivalently, a_n equals the number of compositions of n into any number of parts, where all the parts are either 1 or 2.) Find a recurrence for a_n.

7. (0) **BONUS: 5 points. Do not attempt the bonus problems until you have finished the rest of the exam.** How many 8 digit numbers (positive integers) are there with the following properties?

(a) The digits come from the set $\{0, 1, 2, \ldots, 9\}$.

(b) The first digit is not 0 (so it really is a true 8-digit number).

(c) No digit appears more than once.

(d) The digits $1, 2, 3, 4$ all are used, and appear in that order (not necessarily consecutively) when the number is read left-to-right.

Example: The numbers 18239764 and 98761234 are good, while the numbers 47612398 and 11223344 are not.