LECTURE 25 – PUISEUX SERIES, NEWTON'S POLYGON, AND ALGEBRAIC GENERATING FUNCTIONS

JAY PANTONE

Let $f(z)$ be an algebraic generating function, so that there exists a polynomial $P(z,y) \in \mathbb{C}[z,y]$
$$P(z,y) = p_0(z)y^d + p_1(z)y^{d-1} + \cdots + p_d(z),$$
such that $P(z, f(z))$.

The solution to the equation is $P(z,y)$ is a set of points in $\mathbb{C} \times \mathbb{C}$ called a *complex algebraic curve*. For all but a finite number of z-values, there are exactly d possible values of y, corresponding to different branches. At z-values z_0 where $p_0(z_0) = 0$, there is a reduction in the number of y-values. Alternatively, at z-values z_0 such that $P(z_0, y)$ has a multiple root, some of the y-values will coincide.

Recall that the discriminant of a polynomial equals zero if and only if the polynomial has distinct roots. To that end, define
$$E[P] = \{z : (\text{discrim}(P, y))(z) = 0\}.$$
The set $E[P]$ is called the *exceptional set*, and it represents the location of the possible singularities of $f(z)$ if $P(z, f(z)) = 0$, because these are exactly the places where it is possible that $p_0(z) = 0$ or $P(z,y)$ has a multiple root. For any $z \notin E[P]$, the Implicit Function Theorem tells us that each solution curve y_j lifts to a locally analytic function $y_j(z)$. The $y_j(z)$ are the branches.

To derive asymptotic behavior of $f(z)$, we must understand the behavior of the d branches at points $z \in E[P]$. When $p_0(z) = 0$, some of the branches become infinite and thus are not analytic. At other z-values in $E[P]$, two or more branches collide, either in a multiple point where the branches are still analytic, or in a branch point where they are no longer analytic.

Consider for example the generating function $f(z)$ for the catalan numbers, which satisfies $P(z, f(z)) = 0$ for
$$P(z,y) = y - 1 - zy^2.$$
The discriminant of P with respect to y is
$$z(1-4z),$$
which gives candidates for $z = 0$ and $z = 1/4$ to be the dominant singularity.[1] At $z = 0$ we have $p_0(z) = 0$, and as the accompanying graphs show [see Maple worksheet] one of the branches explodes at $z = 0$. We also can clearly see the branch cut occurring at $z = 1/4$ where the branches collide and are no longer analytic.

[1] It is known the algebraic formal power series are analytic at the origin, precluding a true singularity, but let's ignore that for the moment.

The Lemma below follows from our discussion.

Lemma 25.1. *Let $f(z)$ be analytic at the origin and satisfy $P(z, f(z)) = 0$ for some polynomial $P(z, y)$. Then, $f(z)$ can be analytically continued along any simple path emanating from the origin that does not cross any point of the exceptional set $E[P]$.*

In fact, this process of analytic continuation is how we evaluate the behavior of the branches near a singularity. Suppose for the moment that $z = 0$ is an exceptional point. (This is without loss of generality because you can translate to make it so.) Suppose that the equation $P(0, y)$ has k equal roots $y_1, \ldots, y_k$. We can again assume $y_i = 0$ by translation. Let R be a punctured disc that does not include any other exceptional points of P. Let $y_1(z), \ldots, y_k(z)$ be the k branches colliding at $(0, 0)$.

Start on the branch $y_1(z)$ at some point z of R. By the Implicit Function Theorem, $y_1(z)$ can be analytically continued around the origin on a circuit that returns to z. When this happens, the analytic continuation will have brought you to another branch, say $y_1^{(1)}(z)$. Repeating this, you will eventually get back to the initial branch in, say, κ iterations. Thus we have explored κ branches

$$y_1(z) = y_1^{(0)}(z), y_1^{(1)}(z), \ldots, y_1^{(\kappa)}(z) = y_1(z).$$

These distinct values are called a cycle, and it follows that $y_1(t^\kappa)$ is an analytic function of t except possibly at 0, where it is still continuous and has value 0. Some complex analysis implies that $y_1(t^\kappa)$ has a convergence power series expansion near 0 of the form

$$y_1(t^\kappa) = \sum_{n \geq 1} c_n t^n.$$

Translating back to z gives

$$y_1(z) = \sum_{n \geq 1} c_n z^{n/\kappa},$$

which has κ different determinations, one for each branch in the cycle. It is again clear that $\kappa = 1$ corresponds to an analytic branch. If $\kappa = k$ then a single cycle accounts for all of the roots that tend to 0. Otherwise, we repeat until we've captured all branches that tend to 0 into their various cycles. Branches that tend to infinity are similarly treated with a change of variable $y \mapsto 1/u$.

Theorem 25.2 (Newton-Puiseux Theorem)**.** *Let $f(z)$ be a branch of an algebraic function $P(z, f(z)) = 0$. In a circular neighborhood of a singularity ζ slit along a ray emanating from ζ, $f(z)$ admits a fractional series expansion (Puiseux expansion) that is locally convergent and of the form*

$$f(z) = \sum_{k \geq k_0} c_k (z - \zeta)^{k/\kappa},$$

for a fixed determination of $(z - \zeta)^{1/\kappa}$, where $k_0 \in \mathbb{Z}$ and $\kappa \in \mathbb{N}_{\geq 1}$, called the "branching type".

Newton's Polygon Method

The argument described above is called a "monodromy argument" because it involves running around singularities. Knowing the theorem, we can approach the calculation in a different manner. Consider again the equation

$$P(z, y) = y - 1 - zy^2 = 0,$$

which defines the Catalan generating function. We know algebraically that the two branches of the solution y are

$$\frac{1-\sqrt{1-4z}}{2z} \qquad \text{and} \qquad \frac{1+\sqrt{1-4z}}{2z}.$$

The first should have a branch cut at $z = 1/4$ with no problems at $z = 0$, while the second should have an infinite singularity (a pole) at $z = 0$ and a branch cut at $z = 1/4$. We now examine the behavior of the two branches at $z = 1/4$.

Translate this singularity to the origin via $z \mapsto 1/4 - Z$. This givaes

$$\widehat{P}(Z, y) = -\frac{1}{4}y^2 + y^2 Z + y - 1.$$

At $Z = 0$, $\widehat{P}(0, y)$ has a double root at $y = 2$. Thus perform the change of variables $y \mapsto Y + 2$ so that this double root occurs at $Y = 0$. We obtain

$$Q(Z, Y) = -\frac{1}{4}Y^2 + ZY^2 + 4ZY + 4Z,$$

which has two branches colliding at $(Z, Y) = (0, 0)$. Our goal is to determine the cycle type of the branches. By the Newton-Puiseux theorem, we can look for solutions of the form

$$Y(Z) = cZ^{\alpha}(1 + o(1)),$$

with $c \neq 0$. Supposing this form, each of the monomials of $Q(Z, Y)$ gives an asymptotic order: respectively,

$$Z^{2\alpha}, \qquad Z^{2\alpha+1}, \qquad Z^{\alpha+1}, \qquad Z^1.$$

Since $Q(Z, Y) = 0$, we must find an α value that yields to monomials of the same asymptotic order. We can then calculate the value of c that makes them cancel. Moreover, the remaining monomials must have a smaller asymptotic order (i.e., larger exponents).

With some inspection, we see that there is only one possibility: when $\alpha = 1/2$, the first and last monomials have the same asymptotic order Z^1, while the other two have smaller asymptotic orders Z^2 and $Z^{3/2}$. Therefore, we are assuming the form $Y(Z) = cZ^{1/2}(1 + o(1))$. To determine c, observe that for the monomials $-Y^2/4$ and Z to actually cancel:

$$-\frac{1}{4}(cZ^{1/2})^2 + 4Z = 0,$$

implying

$$-\frac{1}{4}c^2 + 4 = 0,$$

so $c = \pm 4$. This gives the dominant terms for two branches colliding at $(0, 0)$:

$$Y(Z) \sim 4Z^{1/2} \qquad \text{and} \qquad Y(Z) \sim -4Z^{1/2}.$$

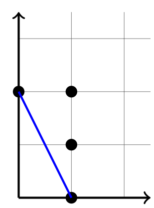
Upon converting back to z and y, this gives

$$f(z) \sim 2 + 4\left(\frac{1}{4} - z\right)^{1/2} \qquad \text{and} \qquad f(z) \sim 2 - 4\left(\frac{1}{4} - z\right)^{1/2}$$

The process can be repeated to arbitrary precision by taking the asymptotic forms for $Y(Z)$ above, subtracting them from $Y(Z)$ and repeating. In other words, perform the analysis on $Y(Z) - 4Z^{1/2}$ and $Y(Z) + 4Z^{1/2}$.

This process is captured with a graphical method called *Newton's Polygon Method*. Suppose
$$Q(Z,Y) = \sum_{j \in J} C_j Z^{a_j} Y^{b_j}.$$
Associate to $Q(Z,Y)$ the set of points (a_j, b_j) in $\mathbb{N} \times \mathbb{N}$. You can verify that finding an α with the restrictions above corresponds to picking the negative reciprocal of a slope between two of the plotted points that are on the left-most convex envelope of the diagram or are to its right. Try these slopes in decreasing asymptotic order looking for one that gives non-zero c. Rinse and repeat!

Example: The polynomial $Q(Z,Y)$ for the Catalan generating functions gives the Newton polynomial below.

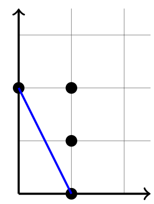


As the following theorem shows, the standard function scale can be applied to Puiseux expansions. (In particular, they are analytic in an indented Δ-domain.

Theorem 25.3. *Let $f(z) = \sum f_n z^n$ be the branch of an algebraic function analytic at 0. Assume that $f(z)$ has a unique dominant singularity at $z = \alpha$. Then, in the non-polar case, the coefficient f_n satisfies*
$$f_n \sim \alpha_1^{-n} \left(\sum_{k \geq k_0} d_k n^{-1-k/\kappa} \right),$$
where $k_0 \in \mathbb{Z}$ and κ is an integer at least 2. In the polar case, $\kappa = 1$ and $k_0 < 0$, the estimate eventually terminates.

With slight modifications, the theorem holds in the case of multiple dominant singularities.

Example: The generating function for unary-binary trees is
$$f(z) = \frac{1 - z - \sqrt{(1+z)(1-3z)}}{2z},$$
which satisfies the minimal polynomial
$$P(z,y) = y - z - zy - zy^2.$$
The discriminant of P with respect to y is
$$(1+z)(1-3z),$$
which indicates potential singularities at $z = 1/3$ and $z = -1$. Making the substitutions $Z = 1/3 - z$ and $Y = y - 1$ gives
$$Q(Z,Y) = -\frac{1}{3}Y^2 + ZY^2 + 3ZY + 3Z.$$
The Newton diagram is



which indicates a dominant asymptotic form $Y \sim cZ^{1/2}$. Solving for c, we find

$$-\frac{1}{3}(cZ^{1/2})^2 + 3Z = 0$$

and so

$$c = \pm 3.$$

Therefore the two branches of $f(z)$ have dominant behavior

$$f(z) \sim 1 + 3\left(\frac{1}{3} - z\right)^{1/2} \qquad \text{and} \qquad f(z) \sim 1 - 3\left(\frac{1}{3} - z\right)^{1/2}$$

near $z = 1/3$. The expansion on the right gives a positive value for the coefficients of the combinatorial series, and so it is the correct one. This computation can be easily carried out by a CAS to arbitrary precision. For example,

$$f(z) \sim 1 - 3d + \frac{9}{2}d^2 - \frac{63}{8}d^3 + \frac{27}{2}d^4 - \frac{2997}{128}d^5 + \frac{81}{2}d^6 + \cdots,$$

where $d = \sqrt{\frac{1}{3} - z}$. The expansion of $f(z)$ near $z = 1/3$ gives immediately the expansion of $[z^n]f(z)$.

Example: The generating function $f(z)$ for bicolored supertrees satisfies $P(z, f(z)) = 0$ for

$$P(z, y) = y^4 - 2y^3 + (1 + 2z)y^2 - 2yz + 4z^3.$$

There are four branches of this function. At $z = 0$, two of them collide at $y = 0$, while two collide at $y = 1$. Of the two that collide at 0, one of them is the branch that gives our combinatorial sequence:

$$2z^2 + 2z^3 + 8z^4 + \cdots.$$

Observing that

$$\operatorname{discrim}(P, y) = 16z^4(16z^2 + 4z - 1)(4z - 1)^3,$$

candidates for the singularities on this branch are

$$z = 0, \frac{1}{4}, \frac{1}{8}(-1 \pm \sqrt{5}).$$

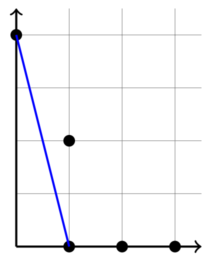
The critical thing to understand is that each of these may be singularities on some branches but not on others, and we need to figure out which are singularities on the branch corresponding to our combinatorial sequence.

In a bit, we'll discuss how to determine which singularities are present on each branch. For now, we'll just assume that $z = 1/4$ is the dominant singularity for our generating function.

To expand $f(z)$ around the singularity at $z = 1/4$, we shift the singularity to the origin, check the y-value, then shift that to the origin. Upon substituting $z = 1/4 - Z$, and then substituting $Z = 0$ and solving for y, we find that all four branches collide at $y = 1/2$. Thus we replace $y = Y + 1/2$ giving

$$Q(Z,Y) = Y^4 - 2ZY^2 - \frac{1}{4}Z + 3Z^2 - 4Z^3,$$

such that the algebraic curve $Q(Z,Y) = 0$ has a singularity at $(Z,Y) = (0,0)$. The Newton polygon is



This indicates a four-cycle on the branches (which we can see from the graphs by very careful inspection). We've already seen how to calculate the Puiseux series by hand (substitute $Y(Z) = cZ^{1/4}$, solve for c, subtract from the original, and repeat). To calculate it with a CAS, simply perform the substitution $Z = d^4$ and find the series expansion with respect to d. We recover four possible expansions:

$$\begin{aligned}
Y_1(d) &= \frac{1}{\sqrt{2}}d + \frac{1}{\sqrt{2}}d^3 - \frac{5}{2\sqrt{2}}d^5 + \frac{5}{2\sqrt{2}}d^7 - \frac{45}{8\sqrt{2}}d^9 + O(d^{11}),\\
Y_2(d) &= \frac{i}{\sqrt{2}}d - \frac{i}{\sqrt{2}}d^3 - \frac{5i}{2\sqrt{2}}d^5 - \frac{5i}{2\sqrt{2}}d^7 - \frac{45i}{8\sqrt{2}}d^9 + O(d^{11}),\\
Y_3(d) &= -\frac{1}{\sqrt{2}}d - \frac{1}{\sqrt{2}}d^3 + \frac{5}{2\sqrt{2}}d^5 - \frac{5}{2\sqrt{2}}d^7 + \frac{45}{8\sqrt{2}}d^9 + O(d^{11}),\\
Y_4(d) &= -\frac{i}{\sqrt{2}}d + \frac{i}{\sqrt{2}}d^3 + \frac{5i}{2\sqrt{2}}d^5 + \frac{5i}{2\sqrt{2}}d^7 + \frac{45i}{8\sqrt{2}}d^9 + O(d^{11}).
\end{aligned}$$

Then, transform Z and Y back to z and y. Only one of these $Y_3(d)$ leads to an expansion that gives positive real coefficients. The corresponding $y_3(z)$ is

$$\begin{aligned}
y_3(z) = \frac{1}{2} &- \frac{1}{\sqrt{2}}\left(\frac{1}{4} - z\right)^{1/4} - \frac{1}{\sqrt{2}}\left(\frac{1}{4} - z\right)^{3/4} + \frac{5}{2\sqrt{2}}\left(\frac{1}{4} - z\right)^{5/4}\\
&- \frac{5}{2\sqrt{2}}\left(\frac{1}{4} - z\right)^{7/4} + \frac{45}{8\sqrt{2}}\left(\frac{1}{4} - z\right)^{9/4} + O\left(\left(\frac{1}{4} - z\right)^{11/4}\right).
\end{aligned}$$

THE ALGEBRAIC COEFFICIENT ASYMPTOTICS (ACA) ALGORITHM

The techniques discussed above, together with rigorous numerical estimates, provide an algorithm to start with the minimal polynomial of a generating function and produce full asymptotic expansion to arbitrary precision.

The two theoretical complications that must be overcome are:

(1) we have to identify the dominant singularities of the generating function, and

(2) we have to determine which asymptotic expansion at each dominant singularity corresponds to the generating function we seek.

This is what is known as a *connection* problem. We are able to solve it in the algebraic case in contrast to the setting of linear differential equations where numerical approximation is the only known recourse.

The algorithm is taken from the textbook (*Analytic Combinatorics*), page 504, which provides a skeleton outline of the procedure.

Step 1: Preparation. Let $f(z)$ be a generating function with minimal polynomial $P(z,y)$. Let $d = \deg_y(P)$. Calculate the exceptional set (adding in 0 if it's not already there)

$$E[P] = \{0\} \cup \{z : (\mathrm{discrim}(P,y))(z) = 0\}.$$

Then calculate the Puiseux expansion of each of the d branches at each $\alpha \in E[P]$. We now have a collection of expansions

$$\{y_{\alpha,j}(z) \;:\; \alpha \in E[P],\; j \in \{1,\ldots,d\}\}.$$

Figure out which of the $y_{0,k}$ corresponds to $f(z)$ by matching initial terms of the expansion at 0 to the known initial terms of the counting sequence.

Step 2: Locating the Dominant Singularity. Partition $E[P]$ into sets $E_0, E_1, \ldots$ by modulus, so that the sets E_i are ordered such that if $i < j$ with $\alpha \in E_i$ and $\beta \in E_j$ then $|\alpha| < |\beta|$. Note that $E_0 = \{0\}$. To find the dominant singularity, we examine each E_i in increasing order until a singularity is located.

To do this, we need to determine a non-zero lower bound δ on the radius of convergence of any local Puiseux expansion at any branch at any point of $E[P]$. (Hint: find the minimal distance between elements of $E[P]$.)

Starting with E_1, examine each E_j in increasing order. For each exceptional point σ_k in E_j, set $\zeta_k = (1 - \delta/2)\sigma_k$. Use resultants to compute a non-zero lower bound η_k on the distance between the branches at $z = \zeta_k$. Then, calculate $f(\zeta_k)$ and $y_{\sigma_k,i}(\zeta_k)$ to check which branch matches the branch $f(z)$ at $z = \zeta_k$. (Calculate each to an accuracy of $\eta_k/4$ to match.) Determine if σ_k is actually a singularity on the appropriate branch (i.e., a pole or a branch point).

Repeat until all dominant singularities are found.

Step 3: Coefficient Expansion. Expand $f(z)$ at each of the dominant singularities and use the standard function scale to translate to an asymptotic expansion of the coefficients.