LECTURE 24 – MULTIPLE SINGULARITIES

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As with meromorphic functions, the main takeaway from this section is: When there are multiple dominant singularities, their contribution to the asymptotic behavior should be added together.

To start with, we must extend the notion of a Δ domain to one that can accommodate multiple singularities on the boundary of the disc of convergence. We call the result a *star domain*. It is really just the intersections of the Δ-domains for each singularity.

[picture goes here, some day]

The theorem is then as follows:

Theorem 24.1 (Master Theorem, multiple dominant singularities)**.** *Let $f(z)$ be analytic in $|z| < \rho$ and have a finite number of singularities on the circle $|z| = \rho$ at points $\zeta_j = \rho e^{i\theta_j}$, for $j = 1, \ldots, r$. Assume that there exists a Δ-domain Δ_0 such that $f(z)$ is analytic in the indented disc*

$$D = \bigcap_{j=1}^{r} (\zeta_j \cdot \Delta_0),$$

with $\zeta \cdot \Delta_0$ the image of Δ_0 by the mapping $z \mapsto \zeta z$.

Assume that there exist r function $\sigma_1, \ldots, \sigma_r$, each a linear combination of elements from the scale $\mathcal{S}$ (defined in previous topic), and a function $\tau \in \mathcal{S}$ such that

$$f(z) = \sigma_j(z/\zeta_j) + O(\tau(z/\zeta_j))$$

as $z \to \zeta_j$ in D. Then, the coefficients of $f(z)$ satisfy

$$f_n = \left(\sum_{j=1}^{r} \zeta_j^{-n} \sigma_{j,n} \right) + O(\rho^{-n} \tau_n^{\star})$$

where each $\sigma_{j,n} = [z^n]\sigma_j(z)$ has its coefficients determine by the previous theorems, and $\tau_n^{\star} = n^{a-1} \log^b(n)$ if $\tau(z) = (1-z)^{-a} \log(1/(1-z))^b$.

The proof is similar to the case of a single dominant singularity, except we must traverse each singularity carefully.

Example: Let $g(z) = \dfrac{e^z}{\sqrt{1-z^2}}$. There are two singularities, $z = \pm 1$. We expand $g(z)$ separately in a neighborhood of each singularity.

$$g(z) \underset{z\to+1}{\sim} \frac{e}{\sqrt{2}}\frac{1}{\sqrt{1-z}}$$
$$g(z) \underset{z\to-1}{\sim} \frac{1}{e\sqrt{2}}\frac{1}{\sqrt{1+z}}$$

Now,

$$[z^n]\frac{e}{\sqrt{2}}\frac{1}{\sqrt{1-z}} \sim \frac{e}{\sqrt{2\pi n}}$$

and

$$[z^n]\frac{1}{e\sqrt{2}}\frac{1}{\sqrt{1+z}} \sim \frac{(-1)^n}{e\sqrt{2\pi n}}.$$

Therefore,

$$[z^n]g(z) \sim \frac{1}{\sqrt{2\pi n}}\left(e + \frac{(-1)^n}{e}\right).$$

Example: Let $\mathcal{F}$ be the class of permutations with cycles of odd length. Then,

$$\mathcal{F} = \textsc{Set}(\textsc{Cyc}_{\text{odd}}(\mathcal{Z}))$$

implying

$$F(z) = \sqrt{\frac{1+z}{1-z}}.$$

It is clear that $F(z)$ has algebraic singularities at $z = \pm 1$, and thus that $F(z)$ is analytic on $\mathbb{C} \setminus (\mathbb{R}_{\geq 1} \cup \mathbb{R}_{\leq -1})$. Therefore we can apply singularity analysis.

As $z \to 1$,

$$F(z) \sim \frac{2^{1/2}}{\sqrt{1-z}} - 2^{-3/2}\sqrt{1-z} + O((1-z)^{3/2}),$$

while as $z \to -1$,

$$F(z) \sim 2^{-1/2}\sqrt{1+z} + O((1+z)^{3/2}).$$

Transferring to coefficients and adding gives

$$[z^n]F(z) = \frac{2^{1/2}}{\sqrt{\pi n}} - \frac{(-1)^n 2^{-3/2}}{\sqrt{\pi n^3}} + O(n^{-5/2}).$$