LECTURE 22 – THE STANDARD FUNCTION SCALE, PART 1

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There are a number of topics that we're skipping over, but I want you to know that they're in the *Analytic Combinatorics* book in case you find that you need them some day.

(1) Periodic and non-periodic fluctuations, conditions for proving periodicity (the Daffodil Lemma).

(2) Localization of zeros and poles, winding number, the Argument Principle, Rouché's Theorem.

(3) Derivation of exponential growth rate from functional equation (e.g., $f(z) = ze^{f(z)}$, $f(z) = z + f(z^2 + z^3)$, $f(z) = (1 - zf(z^2))^{-1}$)

In this section we begin to move past the simple realm of polar singularities into more general territory. Still the basic principle holds that *the asymptotic expansion of a function near its singularities* and *the asymptotic behavior of the function's coefficients* are deeply intertwined.

Our goal, now, is to analyze the asymptotic behavior of sequences coming from a broader class of generating functions with "algebraic-logarithmic" singularities. To do this, we need to central types of theorems.

The first type are *standard function scale* results that let us translate the dominant terms in the asymptotic expansion of a function to the dominant terms in the asymptotic behavior of the sequence, while the second type are *transfer theorems* that let us translate the corresponding error terms.

For example[1], consider a function $f(z)$ such that

$$f(z) \underset{z \to 1}{\sim} \frac{1}{(1-z)^{3/2}} + O\left((1-z)^{-1/2}\right).$$

We shall see that if $\{f_n\}$ is the power series expansion of $f(z)$ at the origin, then the asymptotic expansion above begets the following asymptotic behavior if $\{f_n\}$:

$$f_n \sim n^{1/2} + O(n^{-1/2}).$$

The correspondence between the first terms comes from the standard function scale, while the correspondence between the second terms comes from a transfer theorem.

BIG-O NOTATION

Big-O notation (and its variants—little-o, big-Omega, etc) give a concise way to describe the growth of a function of a sequence.

[1] Postponing just one more time the formal definition of $O(\cdot)$.

Loosely, think of Big-O as $\leq$, and little-o as $<$. Formally, we say

$$f(z) = O(g(z))$$

as $z \to z_0$ if there is a positive constant M and a neighborhood of z_0 such that

$$|f(z)| \leq M|g(z)|$$

for all z in the neighborhood. When $z_0 = \infty$, change "neighborhood of z_0" to "sufficiently large". In the same way, we say

$$a(n) = O(b(n))$$

as $n \to \infty$ if there exists a positive constant M such that for n sufficiently large

$$|a(n)| \leq M|b(n)|.$$

When the second function ($g(z)$ or $b(n)$) is not zero in the limit, these conditions are equivalent to

$$\limsup_{z \to z_0} \left| \frac{f(z)}{g(z)} \right| < \infty \quad \text{and} \quad \limsup_{n \to \infty} \left| \frac{f(n)}{g(n)} \right| < \infty.$$

Really though, $O(g(z))$ and $O(b(n))$ are *equivalence classes*, and we should be saying $f(z) \in O(g(z))$—but *c'est la vie*.

While $f(z) = O(g(z))$ roughly means "$f(z)$ is on the order of $g(z)$ or smaller", little-o notation is stronger. When one write $f(z) = o(g(z))$, they mean "$f(z)$ is of a strictly smaller order than $g(z)$". Formally, $f(z) = O(g(z))$ as $z \to z_0$ if

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = 0.$$

Similar adjustments account for $z \to \infty$ and for the case of sequences.

There are other asymptotic measures as well.

- We say $f(z) = \Omega(g(z))$ if $f(z)$ is asymptotically at least as large as $g(z)$ (up to constant multipliers).
- We say $f(z) = \omega(g(z))$ if $f(z)$ is asymptotically strictly larger than $g(z)$.
- We say $f(z) = \Theta(g(z))$ if $f(z)$ is asymptotically exactly as large as $g(z)$ (up to constant multipliers).

Examples:

- As $x \to 0$, $e^x = 1 + x + O(x^2)$
- As $x \to 0$, $e^x = 1 + x + o(x^{3/2})$
- As $n \to \infty$, $n^3 + 2n^2 + 1 = O(n^{100})$
- As $n \to \infty$, $n^3 + 2n^2 + 1 = O(n^3)$
- As $n \to \infty$, $n^3 + 2n^2 + 1 = o(n^{100})$
- As $n \to \infty$, $n^3 + 2n^2 + 1 = \Theta(n^3)$
- As $n \to \infty$, $n^2 = O(10^{10^{10}} n^2)$ and $n^2 = \omega(10^{10^{10}} n^{1.999})$

In particular, as a function approaches a singularity, its growth is dominated by the additive component that goes to infinity fastest.

THE STANDARD FUNCTION SCALE

Our first step is to find exactly the asymptotic expansions of functions whose expansion near a singularity ζ has the form

$$f(z) \underset{z\to\zeta}{\sim} \left(1-\frac{z}{\zeta}\right)^{-\alpha}\left(\log\left(\frac{1}{1-\frac{z}{\zeta}}\right)\right)^{\beta}.$$

We shall see that this uniformly gives rise to sequences with asymptotic behavior

$$f_n \sim C\cdot\zeta^{-n}n^{\alpha-1}\log(n)^{B},$$

where $B=\beta$ or $B=\beta-1$.

As before, we consider only functions whose dominant singularity is at $z=1$, which causes no loss of generality.

Example: Before we dive in, we present one more example of the power of the forthcoming methods. The class of labeled 2-regular graphs is constructed by

$$\mathcal{R}=\textsc{Set}(\text{U}\textsc{Cyc}_{\geq 3}(\mathcal{Z}))$$

and so it has the exponential generating function

$$R(z)=\exp\left(\frac{1}{2}\left(\log(1-z)^{-1}-z-\frac{z^2}{2}\right)\right)=\frac{e^{-z/2-z^2/4}}{\sqrt{1-z}}.$$

First, we note that the expansion of $e^{-z/2-z^2/4}$ around $z=1$ is

$$e^{-z/2-z^2/4}=e^{-3/4}-e^{-3/4}(z-1)+\frac{e^{-3/4}}{4}(z-1)^2+\frac{e^{-3/4}}{12}(z-1)^3+\cdots.$$

Therefore, near $z=1$,

$$e^{-z/2-z^2/4}=e^{-3/4}+O(1-z).$$

Thus,

$$\frac{e^{-z/2-z^2/4}}{\sqrt{z-1}}=\frac{e^{-3/4}}{\sqrt{1-z}}+O\left(\sqrt{1-z}\right).$$

Our forthcoming work will tell us immediately that

$$R_n\sim\frac{e^{-3/4}}{\sqrt{\pi n}}+O(n^{-3/2}),$$

and more complete asymptotic expansions can be found starting with, for example,

$$e^{-z/2-z^2/4}=e^{-3/4}-e^{-3/4}(z-1)+\frac{e^{-3/4}}{4}(z-1)^2+O((1-z)^3).$$

Polynomial Factors. We have already made a similar calculation for rational generating functions. If $f(z) = (1-z)^{-\alpha}$ for a positive integer α, then by Newton's Generalized Binomial Theorem,

$$\begin{aligned}[z^n]f(z) &= (-1)^n \binom{-\alpha}{n} \\ &= \binom{n+\alpha-1}{n} \\ &= \frac{(n+\alpha-1)(n+\alpha-2)\cdots(n+1)}{(\alpha-1)!} \\ &\sim \frac{n^{\alpha-1}}{(\alpha-1)!}.\end{aligned}$$

What we seek is a method that works for all α and gives an asymptotic expansion to arbitrary accuracy. For this we turn to complex analysis.

Theorem 22.1. *Let α be an arbitrary complex number. The coefficient of z^n in $f(z) = (1-z)^{-\alpha}$ admits, for large n, a complex asymptotic expansion in descending powers of n,*

$$[z^n]f(z) \sim \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \sum_{k=1}^{\infty} \frac{e_k(\alpha)}{n^k}\right),$$

where $e_k(\alpha)$ is a computable polynomial in α of degree $2k$, divisible by $\alpha(\alpha-1)\cdots(\alpha-k)$, and where $1/\Gamma(\alpha)$ is understood to vanish when $\alpha \in \mathbb{Z}_{\leq 0}$. In particular,

$$e_k(\alpha) = \sum_{\ell=k}^{2k} (-1)^{\ell} \lambda_{k,\ell} (\alpha-1)\cdots(\alpha-\ell),$$

where $\lambda_{\ell,k} = [v^k t^{\ell}]e^t(1+vt)^{-1-1/v}$. For example,

$$[z^n]f(z) = \frac{n^{\alpha-1}}{\Gamma(\alpha)}\left(1 + \frac{\alpha(\alpha-1)}{2n} + \frac{\alpha(\alpha-1)(\alpha-2)(3\alpha-1)}{24n^2} + O\left(\frac{1}{n^3}\right)\right).$$

Proof. The main strategy is to use Cauchy's coefficient formula and integrate around a particular contour (a *Hankel contour*, to be precise), that approaches the singularity at $z = 1$ closely before traveling away from it. Additionally, a change of variables $z = 1 + t/n$ and some precise estimates arest needed. See the textbook for the gory details. □