LECTURE 21 – RATIONAL AND MEROMORPHIC FUNCTIONS

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The Second Principle of Analytic Combinatorics says that the sub-exponential component of the asymptotic behavior of a sequence is determined by the *nature* of the generating functions dominant singularity. Unlike the First Principle, which relies on the very concrete notion of the *location* of the dominant singularities, the nature of a singularity is much more nuanced.

We start by examining the case of meromorphic functions, paying particular attention to rational functions. In this situation, all singularities are poles and their nature is fully describable by the multiplicity of the pole.

Rational Functions

Recall our theorem from earlier in the course about coefficient extraction from rational generating functions.

Theorem 21.1. *For any rational generating function $f(z) = P(z)/Q(z)$, (without loss of generality, assume $\deg(P) < \deg(Q)$) the coefficient sequence $\{a_n\}_{n\geq 0}$ can be written in the form*

$$a_n = \sum_{i=1}^{N} P_i(n)\mu_i^n.$$

More particularly, if $f(z)$ can be expanded as

$$f(z) = \sum_{i=1}^{N} \frac{c_i}{(z-\alpha_i)^{r_i}} = \sum_{i=1}^{N} \frac{c_i}{(-\alpha_i)^{r_i}} \frac{1}{(1-\frac{z}{\alpha_i})^{r_i}},$$

then,

$$P_i(n) = \frac{c_i}{(-\alpha_i)^{r_i}} \binom{r_i+n-1}{r_i-1}$$

and

$$\mu_i = \alpha_i^{-1}.$$

This can be easily converted to an asymptotic expansion. Note that

$$\binom{r_i+n-1}{r_i-1} = \frac{(n+r_i-1)(n+r_i-2)\cdots(n+1)}{(r_i-1)!} \sim \frac{n^{r_i-1}}{(r_i-1)!}.$$

Therefore, in the case where $f(z)$ has a unique dominant singularity at α of the form

$$f(z) \underset{z\to\alpha}{\sim} \frac{c}{(z-\alpha)^r},$$

the asymptotic behavior of the sequence is

$$a_n \sim \frac{c}{(-\alpha)^r (r-1)!} n^{r-1} \left(\frac{1}{\alpha}\right)^n.$$

For convenience, we note that the alternate form

$$f(z) \underset{z\to\alpha}{\sim} \frac{c}{(1-z/\alpha)^r},$$

leads to

$$a_n \sim \frac{c}{(r-1)!} n^{r-1} \left(\frac{1}{\alpha}\right)^n.$$

Thankfully, the constant c can be computed without the need for a partial fraction decomposition. If

$$f(z) \underset{z\to\alpha}{\sim} \frac{c}{(z-\alpha)^r},$$

then

$$c = \lim_{z\to\alpha} (z-\alpha)^r f(z).$$

Example: Suppose we are given the generating function

$$f(z) = \frac{1}{(1-z^3)^2(1-z^2)^3(1-z^2/2)}.$$

How can we extract the dominant asymptotic behavior with minimal effort?

First we examine the singularities. The $(1-z^3)^2$ creates singularities at 1, ω, and ω^2 (where ω is a cube root of unity), each of multiplicity 2. The $(1-z^2)^3$ creates singularities at 1 and -1, each of multiplicity 3. The $(1-z^2/2)$ creates singularities at $\pm\sqrt{2}$, each of multiplicity 1.

Overall we have singularities at 1 (multiplicity 5), -1 (multiplicity 3), ω and ω^2 (multiplicity 2), and $\pm\sqrt{2}$ (multiplicity 1).

The dominant singularities are 1 and -1, but since the multiplicity of the pole at 1 is higher than the multiplicity of the pole at -1, only the singularity at 1 needs to be considered.

Since

$$\frac{(1-z)^5}{(1-z^3)^2(1-z^2)^3(1-z^2/2)} = \frac{1}{(1+z+z^2)^2(1+z)^3(1-z^2/2)},$$

we have

$$\lim_{z\to 1} \frac{(1-z)^5}{(1-z^3)^2(1-z^2)^3(1-z^2/2)} = \frac{1}{9\cdot 8\cdot (1/2)} = \frac{1}{36}.$$

Therefore,

$$a_n \sim \frac{1}{36\cdot 4!} n^4 = \frac{n^4}{864}.$$

Example: Let $\mathcal{P}^T$ be the class of integer partitions where the parts are restricted to lie in the *finite set* T. We've seen that the generating function for the class is

$$P^T(z) = \prod_{\omega \in T} \frac{1}{1-z^\omega}.$$

We assume going forward that $\gcd(T) = 1$ (otherwise, replace $z^{\gcd(T)}$ by z everywhere). We are going to see how to extract the asymptotic behavior of $P^T(z)$ in a uniform way.

First we'll look at the case $T = \{1, 2, \ldots, r\}$. For example, the $r = 4$ case is

$$P^{\{1,2,3,4\}} = \frac{1}{(1-z)(1-z^2)(1-z^3)(1-z^4)}.$$

The roots of the denominator are all roots of unity. In the example above, the root 1 occurs with multiplicity 4, the root -1 occurs with multiplicity 2, and the roots ω, ω^2, and $\pm i$ occur with multiplicity 1. For $T = \{1, 2, \ldots, r\}$ it holds in general that the root 1 occurs with multiplicity r and all other roots occur with multiplicity less than r. Actually, this holds for general T by the assumption that $\gcd(T) = 1$.

Therefore, to find the asymptotic behavior of the sequence, we only need to consider the expansion at $z = 1$. Note that

$$P^{\{1,2,\ldots,r\}}(z) \underset{z\to 1}{\sim} \frac{1}{r!}\frac{1}{(1-z)^r}.$$

Hence,

$$p_n^{\{1,2,\ldots,r\}} \sim \frac{1}{r!(r-1)!} n^{r-1}.$$

For more general T with $|T| = r$ (still assuming $\gcd(T) = 1$),

$$P^{\{1,2,\ldots,r\}}(z) \underset{z\to 1}{\sim} \frac{1}{\tau}\frac{1}{(1-z)^r},$$

where

$$\tau = \prod_{\omega \in T} \omega.$$

Suppose for example we want to calculate the asymptotic estimate of the number of ways to make n cents using pennies, nickels, dimes, and quarters. The generating function is

$$\frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})},$$

so $r = 4$ and $\tau = 1 \cdot 5 \cdot 10 \cdot 25 = 1250$. Therefore,

$$p_n^{\{1,5,10,25\}} \sim \frac{1}{1250 \cdot 3!} n^3 = \frac{n^3}{7500}.$$

MEROMORPHIC FUNCTIONS

While all of the examples above involved rational functions, the results carry over to the broader class of functions that are meromorphic at all of their dominant singularities. Although the exact coefficient extraction formula no longer holds, the asymptotic formula does.

Theorem 21.2. *Let $f(z)$ be a function that is meromorphic at all points of the closed disk $|z| \leq R$, with poles at points $\alpha_1, \alpha_2, \ldots, \alpha_m$. Assume that $f(z)$ is analytic on $|z| = R$ and $z = 0$. Then, there exist polynomials such that*

$$f_n = [z^n]f(z) = \sum_{j=1}^{m} \Pi_j(n)\alpha_j^{-n} + O(R^{-n}),$$

and $\deg(\Pi_j)$ *is equal to one less than the order of the pole at α_j.*

Proof. At any pole α we can expand

$$f(z) = \sum_{k \geq -M} a_{\alpha,k}(z-\alpha)^k.$$

Let $S_\alpha(z)$ be the non-analytic terms

$$S_\alpha(z) = \frac{a_{\alpha,-M}}{(z-\alpha)^M} + \cdots + \frac{a_{\alpha,-1}}{z-\alpha},$$

and let $H_\alpha(z)$ be the rest

$$H_\alpha(z) = a_{\alpha,0} + a_{\alpha,1}(z-\alpha) + \cdots,$$

so that

$$f(z) = S_\alpha(z) + H_\alpha(z).$$

Let $S(z)$ be the sum of all $S_\alpha(z)$, so that $f(z) - S(z)$ is analytic in $|z| \leq R$. Now, $S(z)$ is rational and so coefficient extraction proceeds as before. Additionally, the nth coefficient of $f(z) - S(z)$ (expanded at the origin) is bounded above by $(R+\epsilon)^{-n}$ (as can be seen by integrating around $|z| = R$ using Cauchy's coefficient formula), i.e., $f(z) - S(z)$ is $O(R^{-n})$. □

The exponential generating function for surjections is $R(z) = (2 - e^z)^{-1}$. This has a singularities at $z = \log(2) + 2\pi i k$. Of these, only $z = \log(2)$ is dominant. The function is meromorphic at $z = \log(2)$ because

$$\lim_{z \to \log(2)} \frac{z - \log(2)}{2 - e^z} = -\frac{1}{2}.$$

Therefore,

$$R(z) \underset{z \to \log(2)}{\sim} -\frac{1}{2} \cdot \frac{1}{z - \log(2)}.$$

This gives an asymptotic estimate of

$$R_n \sim \frac{n!}{2\log(2)} \cdot \left(\frac{1}{\log(2)}\right)^n$$

Example:

The exponential generating function for alignments (sequences of cycles) is

$$O(z) = \frac{1}{1 - \log(1/(1-z))}.$$

Using the methods above we find

$$O_n \sim \frac{n!}{e\left(1 - \frac{1}{e}\right)} \cdot \left(\frac{1}{1 - \frac{1}{e}}\right)^n.$$

(Note that because the pole occurs at a smaller modulus than the logarithmic singularity, we can treat this with the same methods!)