LECTURE 20 – BASIC COMPLEX ANALYSIS, SINGULARITIES, AND EXPONENTIAL GROWTH

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Overview

The generating function for the catalan numbers is $f(z) = \dfrac{1-\sqrt{1-4z}}{2z}$. So far, we've been thinking of this as a purely formal algebraic object that encodes the coefficients of the Taylor series at $z = 0$. In what follows, we treat generating functions as *analytic objects*. In fact, if we think of $f(z)$ as a complex function, we can recover a tremendous amount of information about the coefficients.

At first this seems counter-intuitive. Sure, it makes sense that $f(0)$ (or, in this case, $\lim_{z\to 0} f(z)$) is the constant term of the series, but what good comes from the knowledge of $f(1/10)$? Or from $f(1)$? Or from $f(1-i)$?

As it turns out, for our purposes the most important feature of a complex function is its set of *singularities*, those points where the function ceases to exist or to be well-defined. We will provide formal definitions later, but for now imagine the functions

$$f(z) = \frac{1}{1-2z} \qquad \text{and} \qquad g(z) = \sqrt{1+z}.$$

The first function will be seen to have a singularity (a *simple pole*) at $z = \frac{1}{2}$. The second is slightly more complicated. In the complex realm, we can take square roots of any number. Recall that in the real domain, we define the square root of a positive number y to be the positive number x such that $y = x^2$. For example, we define $\sqrt{9} = 3$, rather than $\sqrt{9} = -3$. This is done to make the square-root function *single-valued*.

The same care must be taken in the complex domain. Let $z = \rho e^{i\theta}$, and define

$$\sqrt{z} = \sqrt{\rho} e^{i\theta/2}.$$

Informally, the square root of a complex number z is found by halving the angle between the vector pointing from the origin to z (as plotted in the plane) with the vector $\langle 1, 0\rangle$, and then taking the square root of the distance from the origin to z.

This informal definition is not well-defined when z lies on the real axis between -1 and $-\infty$, as "halving the angle" could either mean $\pi \to \pi/2$ or $-\pi \to -\pi/2$. These two possible definitions for $\sqrt{1+z}$ are called the two *branches* of the function. It follows that there is no way to define $\sqrt{1+z}$ continuous in a domain containing $z=-1$ in its interior. Accordingly, we omit the real ray $(-\infty,-1)$ from the domain of definition of $g(z)$. This is called a *branch cut*, while the point $z=-1$ is called a *branch point*. Similar issues arise, for example, with the logarithmic function.

Analytic Functions

Definition: A function $f(z)$ defined over a region Ω is said to be *analytic at* $z_0 \in \Omega$ if there exists an open disc $D \subset \Omega$ centered at z_0 such that $f(z)$ can be represented as a convergent power series expansion centered at z_0 for all $z \in D$, i.e.,

$$f(z) = \sum_{n\geq 0} c_n (z-z_0)^n.$$

The function $f(z)$ is *analytic over* Ω if it is analytic at every point in Ω.

This is a familiar notion from calculus: the *disc of convergence* is exactly the disc such that the series expansion for $f(z)$ is convergent inside the disc and divergent outside the disc. The radius of the disc of convergence is the *radius of convergence*.

Definition: A function f over a region Ω is *complex-differentiable* or *holomorphic* at $z_0 \in \Omega$ if

$$\lim_{\delta->0} \frac{f(z_0+\delta)-f(z_0)}{\delta}$$

exists, where the limit is taken over complex δ. The value of the limit is denoted $f'(z_0)$. The function $f(z)$ is said to be *differentiable in* Ω if it is differentiable at every point in Ω. The usual properties of derivatives of sums, products, quotients, and compositions of functions carry through to the complex domain.

Theorem 20.1 (Equivalence Theorem)**.** *A functions is analytic in a region Ω if and only if it is complex-differentiable in Ω.*

Meromorphic Functions

Definition: A function $h(z)$ is *meromorphic at* z_0 if for z in a punctured neighborhood of z_0, $h(z)$ can be represented as $f(z)/g(z)$ for functions $f(z)$ and $g(z)$ that are analytic at z_0.

A function $h(z)$ that is meromorphic at z_0 can be represented as

$$h(z) = \sum_{n\geq -M} h_n (z-z_0)^n,$$

where $h_{-M} \neq 0$. If $M \geq -1$, then $h(z)$ is said ti have a *pole of order* M at z_0. The *residue of* $h(z)$ *at* $z = z_0$ is

$$\text{Res}[h(z); z=z_0] = h_{-1}.$$

INTEGRALS AND RESIDUES

Definitions: Two paths in a region are said to be *homotopic* if one can be continuously deformed into the other, while staying in Ω. A path can be thought of as a continuous map $\gamma : [0,1] \to \Omega$. A *closed path* is one for which $\gamma(0) = \gamma(1)$. A *simple path* is one for which γ is injective. A closed path is a *loop* if it is homotopic to a single point.

Integrals of complex functions over paths can be taken in the usual way:

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt.$$

Theorem 20.2 (Null Integral Property)**.** *Let f be analytic in Ω and let λ be a simple loop in Ω. Then, $\int_\gamma f(z)dz = 0$. Equivalently, the integral over γ is equal to the integral over γ' if they are homotopic.*

The calculus of complex functions is much deeper than that of real functions in part because of the following reason. Global properties of functions are often mere consequences of local properties of functions at certain points. The simplest example of this behavior is the Cauchy Residue Theorem.

Theorem 20.3 (Cauchy Residue Theorem)**.** *Let $h(z)$ be meromorphic in Ω and let λ be a positively oriented (counterclockwise) simple loop in Ω along which $h(z)$ is analytic. Then,*

$$\frac{1}{2\pi i}\int_\lambda h(z)dz = \sum_s \mathrm{Res}[h(z); z = s],$$

where the sum is over all poles s of $h(z)$ enclosed by λ.

Cauchy's Residue Theorem leads almost immediately to the following theorem, whose combinatorial use is self-evident.

Theorem 20.4 (Cauchy's Coefficient Formula)**.** *Let $f(z)$ be analytic in a region Ω containing 0 and let λ be a simple loop around 0 in Ω that is positively oriented. Then, the coefficient $[z^n]f(z)$ can be written as*

$$[z^n]f(z) = \frac{1}{2\pi i}\int_\lambda \frac{f(z)}{z^{n+1}}dz.$$

Proof. Using Cauchy's Residue Theorem:

$$\frac{1}{2\pi i}\int_\lambda \frac{f(z)}{z^{n+1}}dx = \mathrm{Res}\left[\frac{f(z)}{z^{n+1}};\ z = 0\right] = [z^n]f(z).$$

□

This remarkable formula allows us to calculate coefficients of the expansion of $f(z)$ around $z = 0$ using values of f very far from 0!

SINGULARITIES

As hinted to earlier, a singularity of a function $f(z)$ is a value $z = z_0$ at which $f(z)$ becomes non-analytic. The function $f(z) = (1-z)^{-1}$ which represents the power series $1 + z + z^2 + \cdots$ is non-analytic at $z = 1$. This can be seen by observing that $|f(z)| \to \infty$ as $z \to 1$. Despite the fact that the radius of convergence of the series in question is 1, nothing prevent us from evaluating $f(z)$ at any points $z \neq 1$ of distance further than 1 from the origin.

This is the idea behind *analytic continuation*. Suppose $f(z)$ is analytic over the interior of a region inside a simple closed curve γ and let z_0 be a point on the boundary of γ. If there exists an analytic function $\widehat{f}(z)$ defined over a region $\widehat{\Omega}$ containing z_0 in the interior and such that $\widehat{f}(z) = f(z)$ in $\widehat{\Omega} \cap \Omega$, then f is *analytically continuable* at z_0 with the *analytic continuation* $\widehat{f}$.

We will now show that $f(z) = (1-z)^{-1}$ is analytically continuable to the region $\mathbb{C} \setminus \{1\}$. Let $z_0 \neq 1$. Then,

$$\begin{aligned}
\frac{1}{1-z} &= \frac{1}{1 - z_0 - (z - z_0)} \\
&= \frac{1}{1-z_0} \frac{1}{1 - \frac{z-z_0}{1-z_0}} \\
&= \frac{1}{1-z_0} \sum_{n \geq 0} \left(\frac{z - z_0}{1 - z_0} \right)^n
\end{aligned}$$

This proves that $f(z)$ is analytic in the disc centered at z_0 with radius $|1 - z_0|$.

Unlike continuations of real functions, analytic continuation of complex functions is essentially unique. Analytic continuation can be carried out along any path that does not intersect a singularity. In fact this is often taken as the definition of a singularity.

Definition: Given a function f defined on the interior of a simple closed curve, a point z_0 on the boundary is a *singularity* of f if f is not analytically continuable at z_0.

So far, we have seen poles as the simplest type of singularity. In fact $f(z) = (1-z)^{-1}$ has as its only singularity a simple pole (that is, a pole of multiplicity 1) at $z = 1$.

As mentioned earlier, a power series is (by definition) analytic in its disc of convergent. By the definition above, this means that there can be no singularity within the disc of convergence. We'll now prove something stronger: there *must* be a singularity on the boundary of the disc of convergence.

Theorem 20.5. *Let $f(z)$ be a function that is analytic at the origin with a Taylor series expansion (at the origin) with radius of convergence R. Then, $f(z)$ must have a singularity on the boundary $|z| = R$ of its disc of convergence.*

Proof. Let

$$f(z) = \sum_{n \geq 0} f_n z^n$$

have radius of convergence R. Suppose toward a contradiction that $f(z)$ is analytic inside the disc $|z| < S$ for some $S > R$. Apply Cauchy's Coefficient Formula while integrating around the circle of radius $r = (R+S)/2$:

$$\begin{aligned} f_n &= \frac{1}{2i\pi} \int_{|z|=r} \frac{f(w)}{w^{n+1}} dw \\ &= \frac{1}{2i\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{(re^{i\theta})^n + 1} ire^{i\theta} d\theta \\ &= \frac{r^{-n}}{2\pi} \int_0^{2\pi} \frac{f(re^{i\theta})}{(e^{i\theta})^n}. \end{aligned}$$

Along the integral, the numerator is bounded (otherwise f would not be analytic for $|z| < S$) and the numerator as modulus 1. Therefore, $f_n \leq Cr^{-n}$ for some constant C, implying that the radius of convergence of $f(z)$ is at least r, which is greater than R. This is a contradiciton and so the theorem is proved. □

When the power series of $f(z)$ at the origin has non-negative coefficients (which is combinatorially quite common), we can prove an important refinement that says that not only does $f(z)$ have a singularity at radius R, but it has one on the positive real line at distance R from the origin.

Theorem 20.6 (Pringsheim's Theorem)**.** *If $f(z)$ is representable at the origin by a series expansion that has non-negative coefficients and radius of convergence R, then the point $z = R$ is a singularity of $f(z)$.*

Proof. Here's the general outline of what we'll prove: We'll show that if $f(z)$ has non-negative coefficients then the expansion of f just to the left of R also has positive coefficients, which will lead to a larger radius of convergence than we started with. The picture is below.

Suppose toward a contradiction that $f(z)$ is analytic at $z = R$. Then, it is representable by a convergent power series expansion at $z = R$ with non-zero radius r:

$$f(z) = \sum_{n\geq 0} a_n (z-R)^n,$$

for $|z - R| < r$. Choose h such that $0 < h < r/3$ and consider the expansion of $f(z)$ around $z_0 = R - h$:

$$f(z) = \sum_{n\geq 0} b_n (z-z_0)^n.$$

Letting the series expansion at the origin be written

$$f(z) = \sum_{n\geq 0} f_n z^n,$$

we have

$$\sum_{n\geq 0} f_n z^n = \sum_{n\geq 0} b_n (z-z_0)^n.$$

Use a change of variable $z \to t + z_0$ to get

$$\sum_{n\geq 0} f_n (t+z_0)^n = \sum_{n\geq 0} b_n t^n.$$

and apply the binomial theorem to see that

$$b_n = \sum_{m\geq n} \binom{m}{n} f_m z_0^{m-n},$$

guaranteeing that $b_n \geq 0$ for all n. Additionally, by design the series

$$f(z) = \sum_{n\geq 0} b_n (z-z_0)^n.$$

converges at $z = R + h$, and thus

$$\begin{aligned} f(R+h) &= \sum_{n\geq 0} b_n (2h)^n \\ &= \sum_{n\geq 0} \left(\sum_{m\geq n} \binom{m}{n} f_m z_0^{m-n} \right) (2h)^n \\ &= \sum_{m\geq 0} f_m \left(\sum_{n\geq 0} \binom{m}{n} (R-h)^{m-n} \right) (2h)^n \\ &= \sum_{m\geq 0} f_m (R+h)^m \end{aligned}$$

Since this convergence, f_m must have order smaller than $(R+h)^{-m}$, contradicting the assumption that the radius of convergence was exactly R. □

Definition: Singularities that lie on the boundary of the disc of convergence are called *dominant singularities*. The rest are *sub-dominant*.

Remark: There may be multiple dominant singularities[1]. Pringsheim's Theorem only tells us that if the coefficients of the power series at zero are non-negative than one dominant singularity can be found along the positive real line. The others may be found anywhere.

Examples:

(1) $\frac{1}{1-z^3}$ has three dominant singularities, at $z = 1$, and at the two cubic roots of unity.

(2) $\frac{1}{1+z^2}$ has two dominant singularities, at $z = \pm i$. This does not violate Pringsheim's Theorem because the coefficients of the power series at zero are not non-negative.

(3) $\frac{e^{-z}}{1-z}$ has only one singularity: $z = 1$, and it is (of course) dominant.

(4) $\frac{1}{2-e^{-z}}$ has simple poles wherever $2 - e^z = 0$, i.e., where $e^z = 2$. This gives singularities at $z = \log(2) + 2ik\pi$. Only the singularity at $z = \log(2)$ is dominant.

(5) In order to define the Catalan GF $f(z) = \frac{1-\sqrt{1-4z}}{2z}$ in a single-valued way we have to make a branch cut along the ray $(1/4, \infty)$. Now, we have potential singularities at $z = 0$ and $z = 1/4$. It turns out that the singularity at zero is *removable*: it is possible to define a new function analytic at zero that agrees with $f(z)$ everywhere else. Therefore the real dominant singularity is at $z = 1/4$.

(6) e^{e^z-1} has no singularities. Functions with this property are called *entire*.

1. The Exponential Growth Formula

The application of complex analysis to the study of sequences is largely an attempt to clarify and systemize the following two principles.

The First Principle of Analytic Combinatorics: The exponential growth rate of a sequence is determined by the *location* of its dominant singularities.

The Second Principle of Analytic Combinatorics: The sub-exponential behavior of a sequence is determined by the *nature* of its dominant singularities.

Definition: We say that the *exponential growth rate* of a sequence $\{a_0, a_1, \ldots\}$ is K if

$$\limsup |a_n|^{1/n} = K.$$

This is both an upper and lower bound because for all $\epsilon > 0$ a sequence with growth rate K satisfies

(1) $|a_n| > (K-\epsilon)^n$ infinitely often,

[1] In fact, there can be uncountably many!

(2) $|a_n| < (K+\epsilon)^n$ almost everywhere.

Common Error: If a sequence $\{a_i\}$ has exponential growth rate K, it *is not necessarily true* that $a_n = O(K^n)$. The correct statement is that $a_n = K^n\theta(n)$ where $\theta(n)$ is some *sub-exponential factor* that satisfies

$$\limsup |\theta(n)|^{1/n} = 1.$$

There are many types of sub-exponential factors. Some are nice:

$$1, \qquad n^2, \qquad n^{-4}, \qquad (\log(n))^2 \log(\log(n)),$$

while some usually present difficulties:

$$e^{\sqrt{n}}, \qquad e^{(\log(n))^2}.$$

Theorem 20.7 (Exponential Growth Formula)**.** *Let $f(z)$ be a generating function whose series expansion at $z = 0$ has radius of convergence R. Then, the exponential growth rate of $\{f_0, f_1, \dots\}$ is $1/R$.*

Proof. For any sufficiently small $\epsilon > 0$, we must have $f_n(R-\epsilon)^n \to 0$ (by the definition of radius of convergence). Therefore, $|f_n|(R-\epsilon)^n < 1$ for sufficiently large n, in which case

$$f_n < \frac{1}{(R-\epsilon)^n}.$$

Thus,

$$|f_n|^{1/n} < \frac{1}{R-\epsilon}$$

almost everywhere. On the other hand, $f_n(R+\epsilon)^n$ must be unbounded, or else the radius of convergence is at least $R+\epsilon/2$. Therefore, $|f_n|(R+\epsilon)^n > 1$ infinitely often, implying that

$$|f_n|^{1/n} > \frac{1}{R+\epsilon}$$

infinitely often. The two conditions proved are equivalent to those stated above for a sequence to have exponential growth rate $1/R$. □

Common Error: For some reason, every calls this Pringsheim's Theorem when they cite it in a paper. It's not.

Example: The Catalan numbers have exponential growth rate 4 because of the $\sqrt{1-4z}$ term.

Example: The EGF for surjections is $(2-e^z)^{-1}$. If r_n is the number of surjections, then the sequence $r_n/n!$ has exponential growth rate $(\log(2))^{-1} \approx 1.443$. Put another way

$$r_n \sim \frac{n!}{(\log(2))^n}\theta(n),$$

where $\theta(n)$ is sub-exponential. Recall that the $\sim$ notation formally means

$$\lim_{n\to\infty} \frac{r_n}{\frac{n!}{(\log(2))^n}\theta(n)} = 1,$$

so that if, for example, $\theta(n) \neq 1$ in the example above, it's wrong to write $r_n \sim n!(\log(n))^{-n}$.

Example: We found the generating function for planar rooted unlabeled unary-binary trees to be

$$f(z) = \frac{1 - z - \sqrt{(1-3z)(1+z)}}{2z}.$$

It follows immediately that the counting sequence has exponential growth rate 3.

Major Caution: When the dominant singularity comes from a pole, you must be absolutely certain that it's not removable. You cannot trust Sage or Maple to simplify correctly.