LECTURE 18 – RESULTANT METHODS

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Although the kernel method is very useful, it has its limitations. In particular, when there are more than two unknown series (e.g., $F(z,u)$, $F(z,0)$, and $F_u(z,0)$) or when the functional equation is not linear in $F(z,u)$, the kernel method is not applicable.

Here we will study a technique that applies to functional equations with a single catalytic variable. The results we quote are deep and we will omit their proofs. We *highly* recommend that any combinatorialist reads the work of Bousquet-Mélou and Jehanne [1], from where these results (largely) originate. While the results are quite technical, they are of tremendous practical use in enumerative combinatorics.

For inspiration, let us return to the question of sorting/generating permutations by stacks. It is still an open question which permutations can be sorted by two stacks in series (i.e., what is the basis of this permutation class), and how many there are. In 1991, Julian West asked the following question:

> Suppose we place two stacks in series and try to sort a permutation π by first passing it entirely through the first stack, and then passing it entirely through the second stack. How many permutations can be sorted in this way?

It is critically important to note that these *West-2-stack-sortable* permutations are a proper subset of the permutations that can be sorted with two stacks in series. (Consider 2341.) Soon after West's conjecture, Zeilberger proved that the generating function $f(z,1)$ for these permutations satisfies the functional equation

$$f(z,u) = \frac{1}{1-zu} + \frac{zu(f(z,1) - uf(z,u))(f(z,1) - f(z,u))}{(1-u)^2}.$$

Zeilberger then used a large computer search to guess an algebraic defining polynomial in $f(z,u)$, z, and u, which he then proved rigorously. While the guessing approach is more generally applicable, we will show in this section that not only are functional equations of this kind universally guaranteed to be algebraic, but that we can solve the functional equations to find them nearly instantly.

Resultants and Determinants

Let K be an algebraically closed field and let $P, Q \in K[z]$ have leading coefficients p and q. For concreteness, we operate in $K = \mathbb{C}$. The *resultant* of P and Q with respect to z is

$$\mathrm{Res}(P,Q,z) = p^{\deg(Q)} q^{\deg(P)} \prod_{(x,y):P(x)=Q(y)=0} (x-y).$$

[1] Bousquet-Mélou, M., and Jehanne, A. Polynomial equations with one catalytic variable, algebraic series and map enumeration. *J. Combin. Theory Ser. B 96*, 5 (2006), 623–672

Though it is not obvious from this definition, the resultant of P and Q is a polynomial of the coefficients of P and Q. To see this, rewrite

$$\mathrm{Res}(P, Q, z) = p^{\deg(Q)} \prod_{x : P(x) = 0} Q(x).$$

From this perspective, it is clear that $\mathrm{Res}(P, Q, z)$ is a polynomial in the coefficients of Q, itself with coefficients coming from the roots of P. By the symmetry of the equation, $\mathrm{Res}(P, Q, z)$ is also a polynomial in the coefficients of P, itself with coefficients coming from the roots of Q. Therefore, in fact, $\mathrm{Res}(P, Q, z)$ is really just a polynomial of the coefficients of P and Q.

In practice, the resultant is calculated by setting up a particular matrix with entries from the coefficients of P and Q (called the *Sylvester matrix*). Then, $\mathrm{Res}(P, Q, z)$ is the determinant of that matrix. However, the initial definition of the resultant as a product of root differences reveals the following fact, which is of critical importance to use in this lecture.

Lemma 18.1. *The resultant* $\mathrm{Res}(P, Q, z) = 0$ *if and only if P and Q have a common root.*

Given a single polynomial P, the *discriminant of P with respect to z* is the resultant of P with its own derivative, i.e.,

$$\mathrm{discrim}(P, z) = \mathrm{Res}(P, P_z, z).$$

This is the same discriminant that plays a role in the quadratic equation, as one can show that

$$\mathrm{discrim}(az^2 + bz + c, z) = b^2 - 4ac.$$

From Lemma 18.1, one finds the following consequence.

Lemma 18.2. *The discriminant* $\mathrm{discrim}(P, z) = 0$ *if and only if P has a multiple root.*

The results of Bousquet-Mélou and Jehanne tell us that we can solve a large swatch of functional equations with a single catalytic variable simply by calculating a series of resultants and discriminants.

Solving Functional Equations

The kernel method as a resultant. We shall start by looking at functional equations solvable via the kernel method, as in the previous lecture. There, we found the generating function $f(z, u)$ that satisfies

$$f(z, u) = 1 + uf(z, u) + \frac{z}{u}\left(f(z, u) - f(z, 0)\right)$$

by first rewriting as

$$K(z, u)f(z, u) = P(f(z, 0), z, u)$$

for $K(z, u) = u - u^2 - z$ (called the kernel of the equation) and $P(x_1, z, u) = u - zx_1$. Note that for any series $U(z)$, $K(z, U(z)) = 0$ if and only if $P(f(z, 0), z, U(z)) = 0$.[2] Therefore, the resultant of K with P with respect to u gives a polynomial $R(x_1, z)$ that equals 0. This

[2] We know by its combinatorial definition that $f(z, U(z)) \neq 0$.

equation is the defining equation for the $f(z,0)$ that was sought. For example, using $K(z,u)$ and $P(x_1,z,u)$ as above,

$$\mathrm{Res}(K,P,u) = -x_1^2z^2 + x_1z - z,$$

implying (after dividing by $-z$), that

$$zf(z,0)^2 - f(z,0) + 1 = 0,$$

from which one recovers the Catalan generating function. Thus applying the resultant automatically executes the kernel method.

Higher degree in $f(z,u)$, two unknown series. The technique does not work when the equation is not linear in $f(z,u)$. Suppose instead that we have a polynomial $P(x_0,x_1,z,u)$ such that

$$0 = P(f(z,u),F_1(z),z,u).$$

Theorem 14 of the aforementioned work of Bousquet-Mélou and Jehanne guarantees that if a root $U(z)$ can be found of $P_{x_0}(f(z,u),F_1(z),z,u)$, then $\mathrm{discrim}(P,x_0)$ has a double u-root. Therefore,

$$\mathrm{discrim}(\mathrm{discrim}(P,x_0),u) = 0.$$

The left-hand side is a polynomial in x_1,z, and it is thus a defining equation for $F_1(z)$ unless the left-hand side is zero. This can happen if $\mathrm{discrim}(P,x_0)$ has repeated factors, so to avoid this problem we must inspect each factor of $\mathrm{discrim}(P,x_0)$ to determine which one is really the defining polynomial $R(x_1,z,u)$ such that $R(F_1(z),z,u) = 0$. With this caveat, the iterated discriminant automatically solves all functional equations of this form.

In particular, we can return to the example given in the introduction. By clearing denominators we obtain a polynomial

$$\begin{aligned} P(x_0,x_1,z,u) = -zu^2(zu-1)x_0^2 + (zu(zu-1)(u+1)x_1 + \\ (zu-1)(u-1)^2)x_0 - zu(zu-1)x_1^2 + (u-1)^2 \end{aligned}$$

such that

$$0 = P(f(z,u),f(z,1),z,u).$$

The discriminant of P with respect to x_0 is

$$\mathrm{discrim}(P,x_0) = (u-1)^2(zu-1)R(x_1,z,u),$$

where R is irreducible. Since $\mathrm{discrim}(P,x_0) = 0$ (guaranteed by a theorem from Bousquet-Mélou and Jehanne), one of the factors above must be the defining polynomial for $F_1(z)$ in terms of z and u. As it clearly can't be $(u-1)$ or $(zu-1)$ (since these don't involve x_1), it must be R. Now, since the iterated discriminant is guaranteed to be zero,

$$0 = \mathrm{discrim}(R,u) = -16z(zx_1+1)^2Q(x_1,z).$$

One of these factors must be the defining polynomial for x_1 in terms of z. It's clearly not $-16z$ nor $zx_1+1)$, and so it must be

$$Q(x_1,z) = z^2x_1^3 + z(3z+2)x_1^2 + (3z^2-14z+1)x_1 + z^2 + 11z - 1.$$

Therefore the defining polynomial for $f(z) = f(z,1)$ (the GF for the set of permutations we are counting) is

$$z^2f(z)^3 + z(3z+2)f(z)^2 + (3z^2-14z+1)f(z) + z^2 + 11z - 1 = 0.$$

Higher degree in $f(z,u)$, more than two unknown series. Things get even more complicated, but the fact remains all such functional equations can be algorithmically solved. We will not discuss this case any more than to say that the iterated discriminants are not guaranteed to be 0 (although there are various related conjectures), and so one must dig a little deeper to find polynomials with shared roots to feed into the resultant process.