LECTURE 17 – THE KERNEL METHOD

JAY PANTONE

The Kernel Method is a technique that allows one to start with a bivariate functional equation for $f(z,u)$ with another unknown of the form $f(z,0)$ (or similar), and solve for $f(z,0)$ directly. Once $f(z,0)$ is in hand, usually the full $f(z,u)$ can be recovered as well.

Let $P(y_0, y_1, z, u)$ be a polynomial with complex coefficients, and suppose $f(z,u)$ satisfies the functional equation

$$f(z,u) = P(f(z,u), f(z,0), z, u).$$

In order for the kernel method to work:

(1) This functional equation must be well-defined, in that $f(z,u)$ doesn't cancel on both sides.

(2) The right-hand side must be *linear* in $f(z,u)$, i.e., $[y_0^k]P = 0$ for all $k \geq 2$.

(3) The coefficient of $f(z,u)$ on the right-hand side is a polynomial in z and u only.

The Kernel Method is executed as follows.

(1) Gather all terms involving $f(z,u)$ to the left-hand side, writing the equation as

$$K(z,u)f(z,u) = Q(f(z,0), z, u).$$

The term $K(z,u)$ is called the *kernel*.

(2) Solve the equation

$$K(z,u) = 0$$

in terms of u. This gives a set of solutions $u_1(z), \ldots, u_k(z)$, such that $K(z, u_i(z)) = 0$.

(3) Substitute $u = u_i(z)$ into both sides:

$$K(z, u_i(z))f(z, u_i(z)) = Q(f(z,0), z, u_i(z)).$$

As the left-hand side is zero by design, we have eliminated $f(z,u)$ and obtained the equation

$$0 = Q(f(z,0), z, u_i(z)),$$

which involves only $f(z,0)$ and z.

(4) Solve for $f(z,0)$ (or, if an explicit solution is unobtainable, be happy with the minimal polynomial found in the last step). You will obtain one solution for each $u_i(z)$, one of which will have the power series expansion desired; test the solutions obtained by calculating power series expansions of each one to determine which it is.

Example: In the previous lecture we found the following function equation for the generating function $f(z,u)$ that tracks states of stack operations, where z marks the number of pops that have been performed and u marks the number of entries in the stack:

$$f(z,u) = 1 + uf(z,u) + \frac{z}{u}(f(z,u) - f(z,0)).$$

To apply the Kernel Method, we first gather all terms involving $f(z,u)$ to the left-hand side:

$$\left(1 - u - \frac{z}{u}\right) f(z,u) = 1 - \frac{z}{u} f(z,0).$$

To make the kernel a polynomial, multiply both sides by u:

$$\left(u - u^2 - z\right) f(z,u) = u - zf(z,0).$$

The kernel is $K(z,u) = -(u^2 - u + z)$. By the quadratic equation, $K(z,u) = 0$ has solutions

$$\frac{1 \pm \sqrt{1-4z}}{2}.$$

Substituting this into both sides gives

$$0 = \frac{1 \pm \sqrt{1-4z}}{2} - zf(z,0),$$

and so

$$f(z,0) = \frac{1 \pm \sqrt{1-4z}}{2z}$$

The series expansion of the '$+$' solution is

$$\frac{1 + \sqrt{1-4z}}{2z} = z^{-1} - 1 - z - 2z^2 - \cdots,$$

while the expansion of the '$-$' solution is

$$\frac{1 - \sqrt{1-4z}}{2z} = 1 + z + 2z^2 + 5z^3 + \cdots.$$

Clearly, the '$-$' solution is $f(z,0)$. Typically, the full $f(z,u)$ is not needed as u was simply a catalytic variable. However, if desired, one can now obtain it. By substituting the solution for $f(z,0)$ into the original functional equation, we see that

$$f(z,u) = 1 + uf(z,u) + \frac{z}{u}\left(f(z,u) - \frac{1 - \sqrt{1-4z}}{2z}\right).$$

This is linear in $f(z,u)$ (of course) and so the solution is easily determined to be

$$f(z,u) = \frac{1 - 2u - \sqrt{1-4z}}{2(z - u + u^2)}.$$

Many more examples of this ilk can be given, but they all follow the same basic procedure.

The Kernel Method and its extensions are a very active area of research. While the restrictions placed on the Kernel Method limit its use, numerous extensions have been recently derived.

(1) When the equation is not linear in $f(z, u)$ but instead quadratic, then Tutte's *quadratic method* can be employed.

(2) If there is still a single catalytic variable but there are more unknowns (e.g., $f(z, 1)$, $f_u(z, 1)$, $f_{uu}(z, 1)$, etc.), then more general *resultant methods* can help.

(3) In more intricate cases (such as more than one catalytic variable), the (rigorous) technique of *guess-and-check* will be helpful.