LECTURE 14 – BIVARIATE GENERATING FUNCTIONS AND SPECIFICATIONS

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Suppose you are given a doubly-indexed sequence $f_{n,k}$ that counts somethings. Examples include words over the alphabet $\{0,1\}$ with n 0's and k 1's, Dyck paths of length n with k peaks, and trees with n internal vertices and k leaves. In order to study such sequences, we introduce the notion of a bivariate generating function.

Definition: Given a doubly-indexed sequence $f_{n,k}$ the *ordinary bivariate generating function* is defined by

$$f(z,u) = \sum_{n,k\geq 0} f_{n,k} z^n u^k$$

and the *exponential bivariate generating function* is defined by

$$f(z,u) = \sum_{n,k\geq 0} f_{n,k} \frac{z^n}{n!} u^k.$$

One can easily imagine other variants of the EGF (e.g., one with terms $f_{n,k}\frac{z^n}{n!}\frac{u^k}{k!}$), but we shall shortly see why the definition given is best suited to our needs.

Example: Define $a_{n,k} = \binom{n}{k}$. Then, the OGF of $\{a_{n,k}\}$ is

$$\begin{aligned} A(z,u) &= \sum_{n,k\geq 0} \binom{n}{k} z^n u^k \\ &= \sum_{n\geq 0} \left(\sum_{k\geq 0} \binom{n}{k} u^k \right) z^n \\ &= \sum_{n\geq 0} (1+u)^n z^n \\ &= \frac{1}{1-z(1+u)}. \end{aligned}$$

This derivation is backward: our real goal is to find the bivariate generating function directly from the combinatorial specification of the object, and then use the generating function to find the closed form of the coefficients (if it exists).

In most cases, bivariate (or multivariate) generating functions are most effectively employed by using one variable (for us, z) to be in some sense the "main variable". That is, it tracks overall size or length of an object. Then, multiple auxiliary variables (for us, u in the bivariate case) can be used to track or "mark" certain properties of an object.

Let us return to the examples cited in the introduction and determine which of them fit this paradigm. The first example defines $f_{n,k}$ to be the number of binary words with n 0's and k 1's. The ordinary generating function that falls from this definition does not have the desirable properties we state above. Instead, redefine $f_{n,k}$ to be the number of binary words of length n with k 1's. Our definition now conforms with the stated preference, and we haven't lost any utility: if you know the length of a binary words and you know the number of 1's, then certainly you know the number of 0's. The third example in the introduction requires a similar alteration.

The second example in the introduction considers Dyck paths of length n with k peaks.[1] Here we already have a definition of the type we want.

With this paradigm in mind, it should now be clear why the exponential generating function is defined as it is. When counting labeled objects, if z tracks the total size of the object then it is still necessary to divide z^n by $n!$. However, the auxiliary variables will often not require this adjustment.

Let us now delve immediately into the symbolic method, then return later to the question of what information about a sequence may be gleaned from a bivariate generating function.

We use the symbolic method as before, with our large variety of constructions, with a new component: markers. A *marker* is formally a neutral object (size 0) that can be adjoined with any construction via a product. By attaching markers to constructions, we can keep track of how often they occur.

Example: Let $\mathcal{C}$ be the class of all compositions. Recall that

$$\mathcal{C} = \textsc{Seq}(\textsc{Seq}_{\geq 1}(\mathcal{Z})).$$

If we wish to track compositions of size n by their number of parts, then we attach a marker μ to each part of the composition, i.e., to $\textsc{Seq}_{\geq 1}(\mathcal{Z}))$. We obtain

$$\mathcal{C} = \textsc{Seq}(\mu\, \textsc{Seq}_{\geq 1}(\mathcal{Z})).$$

We convert the specification in the usual way with the added rule $\mu \mapsto u$.[2] This gives

$$C(z,u) = \frac{1}{1 - u\dfrac{z}{1-z}} = \frac{1-z}{1-(u+1)z}.$$

Now, magically, this is the bivariate OGF for compositions where z tracks size and u tracks number of parts. Just once, let us expand the symbolic construction given above so that

[1] A peak is an up-step followed immediately by a down-step.

[2] The marker μ must have size 0 in the specification in order to not change the size of a composition. Then, when converted to a generating function, each occurrence of μ becomes a power of u.

we see why this works. Let $\mathcal{I} = \textsc{Seq}_{\geq 1}(\mathcal{Z})$. Then,

$$\begin{aligned}
\mathcal{C} &= \textsc{Seq}(\mu\,\mathcal{I}) \\
&= 1 + (\mu\,\mathcal{I}) + (\mu\,\mathcal{I}) \times (\mu\,\mathcal{I}) + (\mu\,\mathcal{I}) \times (\mu\,\mathcal{I}) \times (\mu\,\mathcal{I}) + \cdots \\
&= 1 + \underbrace{\mu\mathcal{I}}_{\substack{\text{one}\\ \text{part}}} + \underbrace{\mu^2(\mathcal{I} \times \mathcal{I})}_{\substack{\text{two}\\ \text{parts}}} + \underbrace{\mu^3(\mathcal{I} \times \mathcal{I} \times \mathcal{I})}_{\substack{\text{three}\\ \text{parts}}} + \cdots .
\end{aligned}$$

For each sequence of length k of objects from $\mathcal{I}$, k occurrences of the μ marker are attached.

Example: Let $\mathcal{C}$ be the class of all compositions tracked by size, number of parts equal to 1, and number of parts equal to 2. Then,

$$\mathcal{C} = \textsc{Seq}(\mu_1\,\mathcal{Z} + \mu_2(\mathcal{Z} \times \mathcal{Z}) + \textsc{Seq}_{\geq 3}(\mathcal{Z}))$$

giving

$$C(z, u_1, u_2) = \frac{1}{1 - u_1 z - u_2 z^2 - \dfrac{z^3}{1-z}}.$$

Example: Let $\mathcal{C}$ be the class of all compositions tracked by size, number of parts, and number of parts equal to 1. Then,

$$\mathcal{C} = \textsc{Seq}(\mu\mu_1\,\mathcal{Z} + \mu\,\textsc{Seq}_{\geq 2}(\mathcal{Z}))$$

giving

$$C(z, u, u_1) = \frac{1}{1 - uu_1 z - u\dfrac{z^2}{1-z}}.$$

Coefficient extraction can yield closed-form solutions for all of the above examples. The number of compositions of n into k parts is seen to be

$$\begin{aligned}
[z^n u^k]\frac{1-z}{1-(1+u)z} &= [z^n u^k]\frac{1}{1-(1+u)z} - [z^n u^k]\frac{z}{1-(1+u)z} \\
&= [z^n u^k]\frac{1}{1-(1+u)z} - [z^{n-1} u^k]\frac{1}{1-(1+u)z} \\
&= [z^n u^k]\sum_{n\geq 0}(1+u)^n z^n - [z^{n-1} u^k]\sum_{n\geq 0}(1+u)^n z^n \\
&= [u^k](1+u)^n - [u^k](1+u)^{n-1} \\
&= \binom{n}{k} - \binom{n-1}{k} \\
&= \binom{n-1}{k-1}.
\end{aligned}$$

We shall see later that more is true. From the bivariate generating function one can derive all of the *moments* of the distribution of compositions by number of parts: the expected value, the variance, etc.

For now we proceed with more symbolic examples.

Example: Recall that the class of partitions can be constructed as

$$\mathcal{P} = \textsc{MSet}(\mathcal{I}),$$

yielding

$$P(z) = \prod_{n=1}^{\infty} \frac{1}{1-z^n} = \exp\left(\sum_{k=1}^{\infty} \frac{z^k}{k(1-z^k)}\right).$$

Analogously, the class of partitions with number of parts marked by μ is

$$\mathcal{P} = \textsc{MSet}(\mu\,\mathcal{I}),$$

and so

$$P(z,u) = \prod_{n=1}^{\infty} \frac{1}{1-uz^n} = \exp\left(\sum_{k=1}^{\infty} \frac{u^k z^k}{k(1-z^k)}\right).$$

Example: Let $\mathcal{G}$ be the class of all plane unlabeled trees. We will count the number of trees in $\mathcal{G}$ of size n where the root node has degree k. To do this, we must mark each child of the root, but not children of non-root nodes. We accomplish this with a two-line construction:

$$\mathcal{G}^\circ = \mathcal{Z} \times \textsc{Seq}(\mu\,\mathcal{G})$$
$$\mathcal{G} = \mathcal{Z} \times \textsc{Seq}(\mathcal{G}).$$

Here, $\mathcal{G}^\circ$ represents the whole tree, while $\mathcal{G}$ represents all subtrees (whose root node is not tracked). From this we translate to generating functions:

$$G(z,u) = \frac{z}{1-uH(z)}$$
$$H(z) = \frac{z}{1-H(z)},$$

which can be solved algebraically for $G(z,u)$.

We can treat one collection of marked constructions with a bit more generality. Let $\mathcal{A}$ be a combinatorial class with OGF $A(z)$. Suppose that we want to count sequences of objects from $\mathcal{A}$, marking only those objects from $\mathcal{A}$ of size r. (This is akin to counting compositions of size n, with k parts of size r. Then, the generating function $B(z,u)$ for this class is

$$B(z,u) = \frac{1}{1-A(z)-(u-1)A_r z^r}.$$

The equivalent formula for multisets of objects from $\mathcal{A}$ is

$$B(z,u) = C(z) \cdot \left(\frac{1-z^r}{1-uz^r}\right)^{A_r},$$

where $C(z)$ is the OGF for $\textsc{MSet}(\mathcal{A})$.

For one last general example, suppose that we want to count multisets of objects from $\mathcal{A}$ in such a way that u tracks the number of *distinct* objects in the multiset. Then,

$$\mathcal{B} = \prod_{\alpha \in \mathcal{A}} (\mathcal{E} + \mu\,\textsc{Seq}_{\geq 1}(\alpha))$$

which gives the OGF

$$B(z) = \prod_{n=1}^{\infty} \left(1 + \frac{uz^n}{1-z^n}\right)^{A_n}.$$

Labeled constructions can be marked and translated to bivariate generating functions in much the same way. We illustrated this with just one example.

Example: Let $\mathcal{P}$ be the class of permutations. Let u track the number of cycles in a permutation. Then,

$$\mathcal{P} = \textsc{Set}(\mu \, \textsc{Cyc}(\mathcal{Z})),$$

and so

$$P(z) = \exp\left(u \log\left(\frac{1}{1-z}\right)\right) = (1-z)^{-u}.$$

Let us refine this slightly. Now, fix r and let u track the number of cycles of length r. Then,

$$\mathcal{P} = \textsc{Set}(\textsc{Cyc}_{\neq r}(\mathcal{Z}) + \mu \, \textsc{Cyc}_{=r}(\mathcal{Z})),$$

giving

$$P(z,u) = \exp\left(\log\left(\frac{1}{1-z}\right) + (u-1)\frac{z^r}{r}\right) = \frac{e^{(u-1)z^r/r}}{1-z}.$$

Let's peek ahead for a moment to the type of analysis that we can perform on BGFs. We know that $n![z^n u^k]P(z,u)$ is the number of permutations of length n with k cycles of length r. Call this number $a_{n,k}$. Now,

$$n![z^n](D_u P)(z,1) = \sum_{k \geq 1} k a_{n,k}.$$

Therefore,

$$[z^n](D_u P)(z,1)$$

is the expected number of cycles of length r in a permutation of length n. Using the $P(z,u)$ above,

$$(D_u P)(z,1) = \frac{1}{r}\frac{z^r}{1-z},$$

which implies that the expected number of cycles of length r in a permutation of length n is $1/r$ for $n \geq r$.

From this simple fact we may derive a number of interesting results. A typical permutation has, informally, a few small cycles and a few large cycles. On average, a permutation of length n has $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \sim \log(n)$ cycles in total.

Moreover, we can return to the prisoner's riddle and compute the probability that a permutation of length 100 has no cycles of length greater than 50 to be

$$1 - \sum_{n=51}^{100} \frac{1}{n} \approx 0.312.$$

The final topic of this lecture concerns markings in recursive specifications.

Example: Let $\mathcal{G}$ be the class of unlabeled, rooted, planar trees. We aim to find $G(z,u)$ where z tracks the total number of nodes and u tracks the number of leaves. Then,

$$\mathcal{G} = \mu\mathcal{Z} + \mathcal{Z} \times \textsc{Seq}_{\geq 1}(\mathcal{G}),$$

which produces the functional equation

$$G(z,u) = zu + \frac{zG(z,u)}{1 - G(z,u)}.$$

This can be solved algebraically using the quadratic equation, but it's a bit messy. Instead, apply Lagrange Inversion to conclude that

$$g_{n,k} = \frac{1}{n}\binom{n}{k}\binom{n-2}{k-1}.$$

These numbers are (by necessity) a refinement of the Catalan numbers, and they are known as the *Narayana numbers*.

Example: This example also deals with trees, but instead of marking the number of leaves suppose that u tracks the *path length* of the tree: the sum of the distances from the root to every other node, where the distance between two nodes is defined to be the number of edges in the shortest path between them. Consider the typical tree construction $\mathcal{G} = \mathcal{Z} \times \textsc{Seq}(\mathcal{G})$. The path length of a tree is equal to the sum of the path lengths of each subtree of the root plus the total number of nodes in the tree minus 1.

This does not lend itself to a typical symbolic construction, so we give ourselves a little bit of flexibility in description. A tree is still a root followed by a sequence of more trees, however we must adjust the u variable accordingly. In order to contribute an extra u for every node other than the root, we simplify replace z by uz in the sequences of subtrees. Symbolically,

$$\mathcal{G} = \mathcal{Z} \times \left[\textsc{Seq}(\mathcal{G})\right]_{z \to uz}.$$

This gives the functional equation

$$G(z,u) = \frac{z}{1 - G(uz,u)}.$$

From this, all analysis can be performed with asymptotic methods.