LECTURE 11 – TWO-LEVEL CONSTRUCTIONS

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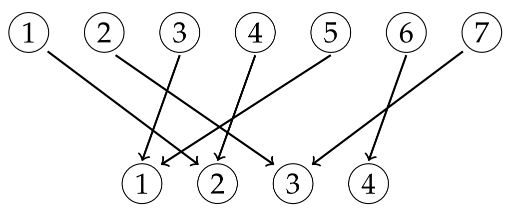
In this lecture we will examine which combinatorial objects we can describe by using iterative (non-recursive) symbolic constructions that are two levels deep. Using SEQ, SET, and CYC there are nine possible constructions, some more natural than others.

Among the labeled classes we'll discus are surjections, set partitions, and permutations.

SURJECTIONS

Definition: A *surjection* of size n is a map $\varphi : [1 \,..\, n] \to [1 \,..\, r]$ such that the image of φ is all of $[1 \,..\, r]$. In other words, for all $j \in [1 \,..\, r]$ there exists some $i \in [1 \,..\, n]$ such that $\phi(i) = j$. Let $\mathcal{R}$ be the class of all surjections (i.e., any r-value is allowed) and let $\mathcal{R}^{(r)}$ be the class of surjections for fixes r. We call these *r-surjections*.

Let φ be a surjection from $[1 \,..\, n]$ to $[1 \,..\, r]$. One way to describe φ is as a word w over the alphabet $\{1 \,..\, r\}$ such that if the ith letter of w equals j, then $\varphi(i) = j$. For the sake of example, let $\varphi : [1 \,..\, 7] \to [1 \,..\, 4]$ be the surjection depicted below.



The word corresponding to this surjection is 2312143. Note that this is an *unlabeled* object. r-surjections of size n are then words of length n over $\{1, \ldots, r\}$ such that *every letter occurs at least once*. This turns out to be a tricky condition to impose via an unlabeled construction.

For example, the case $r = 2$ corresponds to the construction

$$\mathcal{R}^{(2)} = 1\,\textsc{Seq}(1)\,2\,\textsc{Seq}(1+2) \cup 2\,\textsc{Seq}(2)\,1\,\textsc{Seq}(1+2)$$

while the case $r = 3$ can be constructed as

$$\begin{aligned}
\mathcal{R}^{(3)} = {} & 1\,\textsc{Seq}(1)\,2\,\textsc{Seq}(1+2)\,3\,\textsc{Seq}(1+2+3) \\
\cup \;\; & 1\,\textsc{Seq}(1)\,3\,\textsc{Seq}(1+3)\,2\,\textsc{Seq}(1+2+3) \\
\cup \;\; & 2\,\textsc{Seq}(2)\,1\,\textsc{Seq}(1+2)\,3\,\textsc{Seq}(1+2+3) \\
\cup \;\; & 2\,\textsc{Seq}(2)\,3\,\textsc{Seq}(2+3)\,1\,\textsc{Seq}(1+2+3) \\
\cup \;\; & 3\,\textsc{Seq}(2)\,1\,\textsc{Seq}(1+3)\,2\,\textsc{Seq}(1+2+3) \\
\cup \;\; & 3\,\textsc{Seq}(2)\,3\,\textsc{Seq}(2+3)\,1\,\textsc{Seq}(1+2+3).
\end{aligned}$$

More generally, let S be the set of permutations of length r. Then,

$$\mathcal{R}^{(r)} = \bigcup_{\pi \in S} \pi(1)\,\textsc{Seq}(\pi(1))\,\pi(2)\,\textsc{Seq}(\pi(1)+\pi(2)) \cdots \pi(r)\,\textsc{Seq}(\pi(1)+\cdots+\pi(r)),$$

leading to the (ordinary!) generating function

$$R^{(r)}(z) = \frac{6z^r}{(1-z)(1-2z)\cdots(1-rz)}.$$

Exercise: How does this relate to the class of set partitions considered in an earlier lecture?

This specification presents two problems. First, it's not elegant, but more importantly it does not give a clear way to compute the OGF for all surjections. Of course, one can simply sum over r to say that

$$R(z) = \sum_{r=1}^{\infty} \frac{(r!)z^r}{\prod_{k=1}^{r}(1-kz)},$$

but this is not of a form that lends itself to asymptotic analysis.

We now change gears and attempt to find a labeled specification for r-surjections. An r-surjection can be viewed as a sequence of non-empty pre-images. The surjection φ in the example above corresponds to the sequence $(\{3,5\},\{1,4\},\{2,7\},\{6\})$. That the pre-images are non-empty forces φ to be a surjection. Each pre-image is itself a set. Therefore, r-surjections are represented by the labeled construction

$$\mathcal{R}^{(r)} = \textsc{Seq}_{=r}(\textsc{Set}_{\geq 1}(\mathcal{Z})).$$

The EGF for $\textsc{Set}(\mathcal{Z})$ is e^z. Therefore, the EGF for $\textsc{Set}_{\geq 1}(\mathcal{Z})$ is $e^z - 1$. Thus, the EGF for r-surjections is

$$R^{(r)}(z) = (e^z - 1)^r.$$

This form does allow for coefficient extraction and asymptotic analysis, as well as an easy transition to the class of all surjections. Indeed,

$$\mathcal{R} = \textsc{Seq}(\textsc{Set}_{\geq 1}(\mathcal{Z}))$$

and so

$$R(z) = \frac{1}{1-(e^z-1)} = \frac{1}{2-e^z}.$$

We will soon see how to infer from this that the number of surjections of size n has the asymptotic form

$$r_n \sim \frac{n!}{2\log(2)}\left(\frac{1}{\log(2)}\right)^n.$$

Exercise: A *double surjection* is one in which every element in the image $[1\,..\,r]$ is mapped two by at least two elements in the domain. Find the EGF for double surjections.

Set Partitions

Let $\mathcal{S}$ be the class of all set partitions, where the size of a partition is the number of elements. Let $\mathcal{S}^{(r)}$ be the class of set partitions into r non-empty blocks. Set partitions are, in many ways, similar to surjections – the major difference is that the order of the pre-images matters in a surjection, while the order of the blocks does not matter in a set partition. This immediately suggests that

$$\mathcal{S}^{(r)} = \text{SET}_{=r}(\text{SET}_{\geq 1}(\mathcal{Z}))$$

yielding

$$S^{(r)}(z) = \frac{1}{r!}(e^z - 1)^r,$$

as well as

$$\mathcal{S} = \text{SET}(\text{SET}_{\geq 1}(\mathcal{Z}))$$

yielding

$$S(z) = e^{e^z - 1}.$$

Although the first generating function makes it clear immediately that

$$s_n^{(r)} = \frac{1}{r!} r_n^{(r)},$$

the asymptotic form of S_n differs from that of R_n. Saddle-point methods can be used to show that

$$s_n \sim n! \frac{e^{e^r - 1}}{r^n \sqrt{2\pi r(r+1)e^r}},$$

where r is implicitly defined by $re^r = n1$. This can also be expressed as

$$\frac{1}{n}\log(s_n) = \log(n) - \log(\log(n)) - 1 + \frac{\log(\log(n))}{\log(n)} + \frac{1}{\log(n)} + O\left(\left(\frac{\log(\log(n))}{\log(n)}\right)^2\right).$$

Example: Let $e_b(z)$ be the truncated exponential function defined by

$$e_b(z) = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^b}{b!}.$$

Then, set partitions with all block sizes of size at most b are constructed via

$$\mathcal{S}^{\{\leq b\}} = \text{SET}(\text{SET}_{[1\,..\,b]}(\mathcal{Z}))$$

with EGF

$$S^{\{\leq b\}}(z) = e^{e_b(z) - 1}.$$

Exercise: What about set partitions with all block sizes at least b?

PERMUTATIONS

We have already seen a labeled symbolic construction for permutations, though we did not call them permutations at the time. A permutation of length n is an ordered sequence of the numbers from 1 to n. This is equivalent to the construction

$$\mathcal{P} = \textsc{Seq}(\mathcal{Z})$$

giving the exponential generating function

$$P(z) = \frac{1}{1-z}.$$

However, permutations are often written in *cycle notation* rather than in the *one-line notation describe above*. The permutation in one-line notation $\pi = 3164527$ has cycle notation $\pi = (1362)(4)(5)(7)$ (it is common to omit the cycles of length 1, writing $\pi = (1362)$). From this perspective, a permutation is a labeled set of cycles (note that the order in which the cycles are written doesn't matter). Therefore,

$$\mathcal{P} = \textsc{Set}(\textsc{Cyc}(\mathcal{Z}))$$

implying

$$P(z) = \exp\left(\log\left(\frac{1}{1-z}\right)\right) = \frac{1}{1-z}.$$

This more "verbose" construction is more useful for describe restricted versions of permutations.

Given the symbolic equivalence $\textsc{Set}(\textsc{Cyc}(\mathcal{Z})) \cong \textsc{Seq}(\mathcal{Z})$, there should be a nice bijection between permutations in cycle notation and permutations in one-line notation (other than the identity). We now sketch the bijection: let π be a permutation in cycle notation. Write π so that each cycle starts with its largest element, and the cycles are written in increasing order by their first entry. Then, drop the parentheses to obtain a permutation in one-line notation. It's not hard to show that the reverse direction can be defined by detecting where parentheses should go and inserting them.

Exercise: Use a generalization of this idea to show that $\textsc{Set} \circ \textsc{Cyc} \cong \textsc{Seq}$. That is, $\textsc{Set}(\textsc{Cyc}(\mathcal{A})) \cong \textsc{Seq}(\mathcal{A})$ for all combinatorial classes $\mathcal{A}$ with $a_0 = 0$. On the level of generating functions, this is the obvious identity

$$\exp(\log(f(z)) = f(z).$$

We now explore a number of variations on restricted permutations.

Example: Let $\mathcal{P}^{(r)}$ be the class of permutations with exactly r cycles. By the symbolic method,

$$\mathcal{P}^{(r)} = \textsc{Set}_{=r}(\textsc{Cyc}(\mathcal{Z})),$$

giving the EGF

$$P^{(r)}(z) = \frac{1}{r!}\left(\log\left(\frac{1}{1-z}\right)\right)^r.$$

Coefficient extraction gives

$$p_n^{(r)} = \frac{n!}{r!}[z^n]\left(\log\left(\frac{1}{1-z}\right)\right)^r.$$

These numbers are called the *Stirling numbers of the first kind* or the *Stirling cycle numbers*, and are denoted $\begin{bmatrix} n \\ r \end{bmatrix}$. These numbers are fundamental to many applications in combinatorics and computer science. Define $p_{n,k}$ to be the probably that a permutation of length n has exactly k cycles, so that

$$p_{n,k} = \frac{1}{n!}\begin{bmatrix} n \\ k \end{bmatrix}.$$

For $n = 100$ and $1 \leq k \leq 10$, we have

k	1	2	3	4	5	6	7	8	9	10
$n_{100,k}$	0.01	0.05	0.12	0.19	0.21	0.17	0.11	0.06	0.03	0.01

From this we that most (about 99.5%) permutations of length 100 have 10 or fewer cycles, and the mean number of cycles is around 5.18.

In our coming discussion on bivariate generating functions, we will show that the mean number of cycles of all permutations of length n is asymptotically equal to $\log(n)$.

Example: This example considers restrictions on the sizes of cycles rather than the number of cycles. A *derangement* is a permutation with no fixed points, i.e, no cycles of length 1. Clearly, the class $\mathcal{D}$ of derangements is

$$\mathcal{D} = \textsc{Set}(\textsc{Cyc}_{\geq 2}(\mathcal{Z}))$$

producing the EGF

$$D(z) = \exp\left(\log\left(\frac{1}{1-z}\right) - z\right) = \frac{\exp(-z)}{1-z}.$$

A simple asymptotic analysis verifies that $n![z^n]D(z) \sim n!e^{-1}$.

More generally, an r-derangement is a permutation with no cycles of length r or less. The corresponding class $\mathcal{D}^{(r)}$ satisfies

$$\mathcal{D}^{(r)} = \textsc{Set}(\textsc{Cyc}_{>r}(\mathcal{Z})),$$

giving the EGF

$$D^{(r)}(z) = \exp\left(\log\left(\frac{1}{1-z}\right) - z - \frac{z^2}{2} - \cdots - \frac{z^r}{r}\right) = \frac{\exp\left(-z - \frac{z^2}{2} - \cdots - \frac{z^r}{r}\right)}{1-z}.$$

The same asymptotic analysis gives

$$n![z^n]D^{(r)} \sim n!e^{-1-\frac{1}{2}-\cdots-\frac{1}{r}}.$$

Example: Derangements are permutations with no short cycles. This example considers the opposite: permutations with no long cycles. An *involution* is a permutation in which all cycles have length 1 or 2. Accordingly the class $\mathcal{I}$ of involutions is

$$\mathcal{I} = \textsc{Set}(\textsc{Cyc}_{1,2}(\mathcal{Z}))$$

giving EGF

$$I(z) = \exp\left(z + \frac{z^2}{2}\right).$$

Coefficient extraction gives

$$I_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(n-2k)!k!2^k}.$$

Similarly, involutions with no fixed points satisfy

$$\mathcal{J} = \textsc{Set}(\textsc{Cyc}_2(\mathcal{Z})),$$

giving

$$J(z) = \exp\left(\frac{z^2}{2}\right),$$

with an even simpler coefficient extraction

$$J_{2n} = (2n-1)(2n-3)\cdots 5 \cdot 3 \cdot 1.$$

More generally, the class of permutations in which all cycles of have length at most r has EGF

$$B^{(r)}(z) = \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^r}{r}\right).$$

Riddle: The (sadistic) warden of a prison gathers up 100 prisoners and offers them a chance to get out of prison early. He sets up the following game. He puts 100 drawers in a line in a room, writes each prisoners name on an index card, and randomly puts one index card in each drawer.

The prisoners are allowed to enter the room one at a time and check 50 drawers, looking for the card with their own name. If all 100 prisoners find the card with their name, then they can all go free. The prisoners are allowed to plan a strategy before-hand, but cannot communicate once the game starts.

The naïve method in which each prisoner randomly picks 50 drawers has a $2^{-100} \approx 0.79 \cdot 10^{-30}$ chance of success. Is there a better way?

Solution: The prisoners assign themselves numbers 1 through 100. The drawers, when read from one side of the room to the other, then represent an unknown permutation of length 100. Each prisoner enters the room and picks a random drawers to open. If that drawer does not contain their name, then they next open the drawer at the index in the permutation corresponding to the name that was in the first draw. For example, say Steve was assigned 18 and Bob was assigned 66. Steve goes into the room, picks a random drawer and opens it, finding that it contains the index card with Bob's name on it. Since Bob was assigned number 66, he then goes to the 66th box in the room and opens that one. The only way that this method can fail is if the permutation induced by the drawers has a cycle of length greater than 50.

The probability that a permutation of length 100 has no cycles of length greater than 50 is

$$[z^{100}] \exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^{50}}{50}\right) \approx 0.31.$$

This gives the prisoners a 31% chance of success!

Example: Let $\mathcal{P}^{(e)}$ be the class of permutations with an even number of cycles. Then,

$$\mathcal{P}^{(e)} = \textsc{Set}_{\text{even}}(\textsc{Cyc}(\mathcal{Z})).$$

What is the generating function for sets of even length? Clearly, if $\mathcal{C}_{\text{even}} = \textsc{Set}_{\text{even}}(\mathcal{A})$ then

$$C(z) = \sum_{n=0}^{\infty} \frac{A(z)^{2n}}{2n} = 1 + \frac{A(z)^2}{2} + \frac{A(z)^4}{4} + \cdots .$$

Let's answer a more general intermediate question. Let

$$f(z) = \sum_{n=0}^{\infty} f_n z^n.$$

What is the generating function for

$$f_{\text{even}}(z) = \sum_{n=0}^{\infty} \left\{ \begin{array}{ll} f_n, & n \text{ is even} \\ 0, & n \text{ is odd} \end{array} \right\} z^n?$$

This is a neat generating trick: in order to eliminate the odd terms and leave the even terms, substitute $-z$ into f and add it back to itself. Then, the even terms have been doubled and the odd terms have finished, so divide by two. Algebraically,

$$f_{\text{even}}(z) = \frac{f(z) + f(-z)}{2}.$$

Similarly,

$$f_{\text{odd}}(z) = \frac{f(z) - f(-z)}{2}.$$

Returning to the previous question, if $\mathcal{C}_{\text{even}} = \textsc{Set}_{\text{even}}(\mathcal{A})$ then

$$C_{\text{even}}(z) = \frac{e^{A(z)} + e^{-A(z)}}{2}.$$

Returning all the way back to the original problem, we now derive

$$P^{(e)} = \frac{\frac{1}{1-z} + (1-z)}{2} = 1 + \frac{z^2}{2(1-z)}.$$

Therefore, $p_n^{(e)} = n!/2$ for $n \geq 2$.

Exercise: This can be proved combinatorially, using the concept of even and odd permutations (in the group-theoretic sense. How?

Example: Next we consider a different question with a more interesting answer. Let $\mathcal{P}^{\{e\}}$ be the class of permutations in which the cycles have even length. The symbolic construction is

$$\mathcal{P}^{\{e\}} = \textsc{Set}(\textsc{Cyc}_{\text{even}}(\mathcal{Z})).$$

The generating function is thus

$$P^{\{e\}}(z) = \exp\left(\frac{\log\left(\frac{1}{1-z}\right) + \log\left(\frac{1}{1+z}\right)}{2} \right) = \frac{1}{\sqrt{1-z^2}}.$$

Other Two-Level Constructions

We've already looked at the two-level constructions $\textsc{Seq} \circ \textsc{Set}$, $\textsc{Set} \circ \textsc{Set}$, and $\textsc{Set} \circ \textsc{Cyc}$. Clearly we can build six more. What do they represent?

SET ∘ SEQ. The class $\mathcal{F} = \textsc{Set}(\textsc{Seq}_{\geq 1}(\mathcal{Z}))$ consists of sets of permutations that have been consistently relabeled. One example of an object in $\mathcal{F}$ of size 9 is

$$\textcircled{8} - \textcircled{4} \qquad \textcircled{6} - \textcircled{7} - \textcircled{1} - \textcircled{2} \qquad \textcircled{3} - \textcircled{9} - \textcircled{5}.$$
$$=$$
$$\textcircled{6} - \textcircled{7} - \textcircled{1} - \textcircled{2} \qquad \textcircled{3} - \textcircled{9} - \textcircled{5} \qquad \textcircled{8} - \textcircled{4}$$

Such objects are called *fragmented permutations*. Their generating function is

$$F(z) = e^{z/(1-z)}.$$

Despite the simple generating function, the asymptotic form of the counting sequence is more complex:

$$\frac{f_n}{n!} \sim \frac{1}{2\sqrt{e\pi}} \frac{(e^2)^{\sqrt{n}}}{n^{3/4}}.$$

The the term $(e^2)^{\sqrt{n}}$ grows larger than any polynomial sequence but slower than any true exponential growth. There is no standardized term for this; some call it *stretched exponential* behavior.

SEQ ∘ SEQ. The class $\mathcal{S} = \textsc{Seq}(\textsc{Seq}_{\geq 1}(\mathcal{Z}))$ can be interpreted as permutations with optional dividing lines between each pair of consecutive entries. Two distinct objects in this class are:

$$\textcircled{8} - \textcircled{4} \;\Big|\; \textcircled{6} - \textcircled{7} - \textcircled{1} - \textcircled{2} \;\Big|\; \textcircled{3} - \textcircled{9} - \textcircled{5}.$$
$$\neq$$
$$\textcircled{6} - \textcircled{7} - \textcircled{1} - \textcircled{2} \;\Big|\; \textcircled{3} - \textcircled{9} - \textcircled{5} \;\Big|\; \textcircled{8} - \textcircled{4}$$

The EGF is

$$S(z) = \frac{1-z}{1-2z}$$

(where have we seen this before?) and the coefficients have the simple closed form $s_n = n!2^{n-1}$.

Compare these last two classes to the duality between compositions and partitions.

SEQ ∘ CYC. The class $\mathcal{A} = \textsc{Seq}(\textsc{Cyc}_{\geq 1}(\mathcal{Z}))$ consists of sequences of directed cycles that are known as *alignments*. Their generating function is

$$A(z) = \frac{1}{1 - \log\left(\dfrac{1}{1-z}\right)}$$

and they have coefficients

$$\frac{a_n}{n!} \sim \frac{1}{e-1}\left(\frac{e}{e-1}\right)^n.$$

Supernecklaces. The remaining two-level constructions are $\textsc{Cyc}(\textsc{Seq}_{\geq 1}(\mathcal{Z}))$, $\textsc{Cyc}(\textsc{Set}_{\geq 1}(\mathcal{Z}))$ and $\textsc{Cyc}(\textsc{Cyc}_{\geq 1}(\mathcal{Z}))$. These are all given the generic name *supernecklaces*. Their generating functions are, of course, all easily found. We leave the details to the reader.