LECTURE 9 – COEFFICIENT EXTRACTION FOR ALGEBRAIC GENERATING FUNCTIONS

JAY PANTONE

In the Lecture 7 we examined how to find a closed-form expression for the coefficients of any rational generating function. Here, we ask the same question for the more general class of algebraic generating functions. Recall that $f(z)$ is algebraic of degree d if there exists polynomial $p_i(z) \in \mathbb{Q}[z]$ such that

$$0 = p_d(z)f^d + p_{d-1}(z)f^{d-1} + \cdots + p_1(z)f + p_0(z).$$

The question of coefficient extraction must be split into two cases.

(1) If we are given a closed-form expression for $f(z)$, or if the minimal polynomial for $f(z)$ is easily solvable to yield a closed-form expression, then coefficient extraction may be carried out using Newton's Generalized Binomial Theorem from the previous lecture.

(2) Otherwise, we use deeper theory to produce a nested-sum expression directly from the minimal polynomial.

Coefficient Extraction from a Closed Form

Recall Newton's Generalized Binomial Theorem: for any real number r,

$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k}y^k = x^r + rx^{r-1}y + \frac{r(r-1)}{2!}x^{r-2}y^2 + \frac{r(r-1)(r-2)}{3!}x^{r-3}y^3 + \cdots .$$

Using this together with various other generating function methods, we can extract coefficients from many explicitly-given algebraic generating functions. One trick that prove handy is:

$$[z^n]f(\alpha \cdot z) = \alpha^n [z^n]f(z).$$

Example: Find the coefficient sequence $\{a_n\}_{n\geq 1}$ for $f(z) = \dfrac{1}{\sqrt{1-4z}}$. First, we use the aforementioned trick to see that

$$[z^n]\frac{1}{\sqrt{1-4z}} = 4^n [z^n]\frac{1}{\sqrt{1-z}}.$$

Now, since $\frac{1}{\sqrt{1-z}} = (1-z)^{-1/2}$, we apply the Binomial Theorem and see

$$\begin{aligned}
[z^n](1-z)^{-1/2} &= [z^n]\sum_{k=0}^{\infty}\binom{-1/2}{k}(-z)^k \\
&= (-1)^n\binom{-1/2}{n} \\
&= (-1)^n\frac{(-1/2)(-3/2)\cdots((1-2n)/2)}{n!} \\
&= \frac{1}{2^n}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
[z^n]\frac{1}{\sqrt{1-4z}} &= 4^n[z^n]\frac{1}{\sqrt{1-z}} \\
&= \frac{4^n}{2^n}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!} \\
&= 2^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!} \\
&= \frac{n!2^n}{n!}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!} \\
&= \frac{2\cdot 4\cdot 6\cdot\cdots\cdot 2n}{n!}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!} \\
&= \binom{2n}{n}.
\end{aligned}$$

Exercise: Use the method described above to prove that

$$[z^n]\frac{1-\sqrt{1-4z}}{2z} = \frac{1}{n+1}\binom{2n}{n}.$$

Coefficient Extraction from a Minimal Polynomial

Suppose that $f(z)$ is algebraic with minimal polynomial

$$0 = p_d(z)f^d + p_{d-1}(z)f^{d-1} + \cdots + p_1(z)f + p_0(z).$$

For $d > 2$ it is often ill-advised or impossible to solve for f explicitly. In order to extract a formula for the coefficients of f, we have to work directly with the minimal polynomial.

Definition: Let D_yP denote the formal derivative of P with respect to the variable y.

Suggested Reading: Banderier, C., and Drmota, M. Coefficients of algebraic functions: formulae and asymptotics. *DMTCS Proceedings 0*, 01 (2013)

Theorem 9.1. *Suppose $f(z)$ is implicitly defined by the equation $f = P(z, f)$, for some polynomial $P(z, y)$ such that $P(0,0) = 0$, $(D_yP)(0,0) = 0$, and $P(z,0) \neq 0$.*[1] *Then, the coefficients $\{f_n\}_{n\geq 1}$*

[1] The condition $P(0,0) = 0$ requires $f(0) = 0$. The condition $(D_yP)(0,0) = 0$ prevents a term cy in $P(z,y)$ for a constant c and this can be forced without loss of generality. The condition $P(z,0) \neq 0$ ensures that

of $f(z)$ are given by

$$f_n = \sum_{m=1}^{\infty} \frac{1}{m}[z^n y^{m-1}]P(z,y)^m.$$

Though it does not look it, this sum is actually finite (i.e., it is zero for sufficiently large m).

Proof. Let $g \in \mathbb{Q}[[z,u]]$ be the power series defined implicitly by the equation

$$g = uP(z,g).$$

Note that now $g(z,1)$ satisfies

$$g(z,1) = uP(z,g(z,1))$$

and therefore $g(z,1) = f(z)$. We now apply Lagrange inversion to $g(z,u)$ treating u as the main variable and z as a fixed parameter. Then, letting $\phi(u) = P(z,g(z,u))$, we have

$$[u^m]g(z,u) = \sum_{m=1}^{\infty} \frac{1}{m}[u^{m-1}]P(z,u)^m$$

so that

$$g(z,u) = \sum_{m=1}^{\infty} \left(\frac{1}{m}[u^{m-1}]P(z,u)^m\right) u^m.$$

Setting $u = 1$, we recover

$$f(z) = \sum_{m=1}^{\infty} \frac{1}{m}[y^{m-1}]P(z,y)^m$$

and thus

$$f_n = \sum_{m=1}^{\infty} \frac{1}{m}[z^n y^{m-1}]P(z,y)^m.$$

□

As this is a finite sum, it does constitute a closed-form formula. However, it can be simplified by using a generalization of the binomial coefficients.

Definition: Let $m = m_1 + m_2 + \cdots + m_k$. The *multinomial coefficient* $\binom{m}{m_1, m_2, \ldots, m_k}$ is the number of ways to select from m distinguishable objects k subsets of sizes m_1, m_2, …, m_k.

Lemma 9.2. *Similarly to the binomial coefficients, we have*

$$\binom{m}{m_1, m_2, \ldots, m_k} = \frac{m!}{m_1! m_2! \cdots m_k!}.$$

Now we can state a generalization of the binomial theorem.

$[y^0]P(z,y)$ is non-zero; if this does not hold then either a y term can be factored from both sides, or else the expression does not implicitly define a formal power series.

Theorem 9.3 (Multinomial Theorem)**.** *The following equality holds:*

$$(u_1 + u_2 + \cdots + u_k)^m = \sum_{\substack{m_1+\cdots+m_k=m \\ m_i \geq 0}} \binom{m}{m_1, m_2, \ldots, m_k} u_1^{m_1} u_2^{m_2} \cdots u_k^{m_k},$$

where the sum is over all weak compositions of m into k parts.

The Multinomial Theorem allows us to give a more concrete expression (albeit, less elegant) for the closed form of the coefficients of an algebraic generating function.

Theorem 9.4. *Suppose $f(z)$ is implicitly defined by the equation $f = P(z, f)$, for some polynomial $P(z, y)$ such that $P(0, 0) = 0$, $(D_y P)(0, 0) = 0$, and $P(z, 0) \neq 0$. Break up $P(z, y)$ into terms and write it as*

$$P(z, y) = \sum_{i=1}^{k} a_i z^{b_i} y^{c_i}.$$

Then

$$f_n = \sum_{m=1}^{\infty} \frac{1}{m} \sum_{\substack{m_1+\cdots+m_k=m \\ b_1 m_1+\cdots+b_k m_k=n \\ c_1 m_1+\cdots+c_k m_k=m-1}} \binom{m}{m_1, m_2, \ldots, m_k} a_1^{m_1} a_2^{m_2} \cdots a_k^{m_k}$$

Proof. The theorem follows from application of the multinomial theorem. First, note that

$$P(z, y)^m = \left(\sum_{i=1}^{k} a_i z^{b_i} y^{c_i} \right)^m = \sum_{m_1+\cdots+m_k=m} \binom{m}{m_1, m_2, \ldots, m_k} (a_1 z^{b_1} y^{c_1})^{m_1} \cdots (a_k z^{b_k} y^{c_k})^{m_k}.$$

Hence,

$$[z^n] P(z, y)^m = \sum_{\substack{m_1+\cdots+m_k=m \\ b_1 m_1+\cdots+b_k m_k=n}} \binom{m}{m_1, m_2, \ldots, m_k} (a_1 y^{c_1})^{m_1} \cdots (a_k y^{c_k})^{m_k},$$

and so

$$[z^n y^{m-1}] P(z, y)^m = \sum_{\substack{m_1+\cdots+m_k=m \\ b_1 m_1+\cdots+b_k m_k=n \\ c_1 m_1+\cdots+c_k m_k=m-1}} \binom{m}{m_1, m_2, \ldots, m_k} a_1^{m_1} \cdots a_k^{m_k}.$$

The statement of the theorem then follows. □

We will compute two examples by hand. In general, the iterated summations provided by this theorem are not expressed as simply as they may be and it's often hard to perform the simplification. Nevertheless, sometimes this is the best one can do (in terms of a closed form).

Example: The Motzkin numbers count many combinatorial objects, including planar rooted unary-binary trees. They are given by a generating function $M(z) = 1 + z + 2z^2 + 4z^3 + \cdots$ that satisfies

$$0 = 1 + (z - 1)M(z) + z^2 M(z)^2.$$

To apply algebraic coefficient extraction, we rearrange

$$M(z) = 1 + zM(z) + z^2 M(z)^2,$$

and set

$$P(z,y) = 1 + zy + z^2y^2.$$

However, this $P(z,y)$ does not satisfy the conditions of the theorem, as $P(0,0) \neq 0$.[2] To remedy this, we shift the sequence of Motzkin numbers over by 1, setting $\widehat{M}(z) = z + z^2 + 2z^3 + 4z^4 + \cdots$. It can be shown that this new $\widehat{M}(z)$ satisfies

$$\widehat{M}(z) = z + z\widehat{M}(z) + z\widehat{M}(z)^2.$$

Now using

$$P(z,y) = z + zy + zy^2,$$

we do indeed meet the conditions $P(0,0) = 0$, $(D_yP)(0,0) = 0$, and $P(z,0) \neq 0$. Set

$$\begin{array}{lll} a_1 = 1 & b_1 = 1 & c_1 = 0 \\ a_2 = 1 & b_2 = 1 & c_2 = 1 \\ a_3 = 1 & b_3 = 1 & c_3 = 2 \end{array}.$$

Then,

$$[z^n]\widehat{M}(z) = \sum_{m\geq 1} \frac{1}{m} \sum_{\substack{m_1+m_2+m_3=m \\ m_1+m_2+m_3=n \\ m_2+2m_3=m-1}} \binom{m}{m_1, m_2, m_3}.$$

Now we must consider what we can conclude about weak compositions of m with these properties. Firstly, we can conclude that $m = n$, which eliminates all terms in the outer summation except for the term with $m = n$. Secondly, by subtracting the third equation from the first, we see that $m_3 = m_1 - 1$. Finally, since $m_1 + m_2 + m_3 = n$, we conclude that $m_2 = n - m_1 - m_3 = n - 2m_1 + 1$. Therefore, once we pick a value for m_1, both m_2 and m_3 are determined. Therefore,

$$[z^n]\widehat{M}(z) = \frac{1}{n} \sum_{m_1\geq 0} \binom{n}{m_1, n-2m_1+1, m_1-1}.$$

To simplify, we make use of a multinomial identity:

$$\begin{aligned} \binom{k_1+k_2+\cdots+k_m}{k_1,k_2,\ldots,k_m} &= \binom{k_1+k_2}{k_2}\binom{k_1+k_2+k_3}{k_3}\cdots\binom{k_1+k_2+\cdots+k_m}{k_m} \\ &= \binom{k_1+k_2}{k_1}\binom{k_1+k_2+k_3}{k_1+k_2}\cdots\binom{k_1+k_2+\cdots+k_m}{k_1+k_2+\cdots+k_{m-1}} \end{aligned}$$

This allows us to write our summation as

$$[z^n]\widehat{M}(z) = \frac{1}{n} \sum_{m_1\geq 0} \binom{2m_1-1}{m}\binom{n}{2m_1-1}.$$

This is a perfectly good answer to the question of coefficient extraction, but in order to match the form most commonly stated for the Motzkin numbers, we apply the binomial identity

$$\binom{n}{k}\binom{n-k}{h} = \binom{n}{h}\binom{n-h}{k},$$

to get

$$[z^n]\widehat{M}(z) = \frac{1}{n} \sum_{m_1\geq 0} \binom{n}{m_1}\binom{n-m_1}{m_1-1}.$$

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[2] We knew this would happen because $M(0) \neq 0$.

To recover the unshifted Motzkin numbers that we originally sought, observe

$$[z^n]M(z) = [z^{n+1}]\widehat{M}(z) = \frac{1}{n+1}\sum_{k\geq 0}\binom{n+1}{k}\binom{n+1-k}{k-1}.$$

Example: Our second example involves a sequence of numbers that counts a combinatorial object known as a *pattern-avoiding permutation class*.[3] The class of permutations that avoid the patterns 4123, 4132, and 4213 has generating function $s(z) = z + 2z^2 + 6z^3 + 21z^4 + \cdots$ satisfying

$$0 = z + (2z-1)s(z) + 2zs(z)^2 + zs(z)^3,$$

and so

$$s(z) = z + 2zs(z) + 2zs(z)^2 + zs(z)^3.$$

Setting

$$P(z,y) = z + 2zy + 2zy^2 + zy^3,$$

we find that the conditions $P(0,0) = 0$, $(D_yP)(0,0) = 0$, and $P(z,0) \neq 0$ are all satisfied.

We now set

$$\begin{array}{lll} a_1 = 1 & b_1 = 1 & c_1 = 0 \\ a_2 = 2 & b_2 = 1 & c_2 = 1 \\ a_3 = 2 & b_3 = 1 & c_3 = 2 \\ a_4 = 1 & b_4 = 1 & c_4 = 3 \end{array}.$$

Therefore,

$$[z^n]s(z) = \sum_{m\geq 1}\frac{1}{m}\sum_{\substack{m_1+m_2+m_3+m_4=m\\ m_1+m_2+m_3+m_4=n\\ m_2+2m_3+3m_4=m-1}}\binom{m}{m_1,m_2,m_3,m_4}2^{m_2}2^{m_3}.$$

As in the previous example, the constraints of the sum imply $m = n$ and so the outer sum vanishes[4]. Moreover, by subtracting the third equation from the first we see that $m_1 = m_3 + 2m_4 + 1$. Lastly, as $m_1 + m_2 + m_3 + m_4 = m$, we have $m_2 = n - 1 - 2m_3 - 3m_4$. Therefore, m_1 and m_2 are determined by m_3 and m_4, and hence

$$\begin{aligned}[z^n]s(z) &= \frac{1}{n}\sum_{m_3\geq 0}\sum_{m_4\geq 0}\binom{m}{m_3+2m_4+1,\, n-1-2m_3-3m_4,\, m_3,\, m_4}2^{n-1-m_3-3m_4}\\ &= \frac{1}{n}\sum_{m_3\geq 0}\sum_{m_4\geq 0}\binom{m_3+m_4}{m_4}\binom{2m_3+3m_4+1}{m_3+2m_4+1}\binom{n}{n-1-2m_3-3m_4}2^{n-1-m_3-3m_4}\\ &= \frac{1}{n}\sum_{m_3\geq 0}\sum_{m_4\geq 0}\binom{m_3+m_4}{m_4}\binom{2m_3+3m_4+1}{m_3+m_4}\binom{n}{2m_3+3m_4+1}2^{n-1-m_3-3m_4}.\end{aligned}$$

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[3]We omit the definition of a permutation class, but the interested read should consult the comprehensive survey: [VATTER, V. Permutation classes. In *Handbook of Combinatorics*, M. Bóna, Ed. CRC Press, 2015].

[4]This does not always happen!