LECTURE 8 – COEFFICIENT EXTRACTION VIA LAGRANGE INVERSION

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Lagrange Inversion is a technique that can be used to extract coefficients from a functional equation for a generating function that takes a particular form.

> **Suggested Reading:** MERLINI, D., SPRUGNOLI, R., AND VERRI, M. C. Lagrange inversion: When and how. *Acta Applicandae Mathematica 94*, 3 (2006), 233–249

Definition: Given a formal power series $f \in \mathbb{Q}[[z]]$, we say that $f^{\langle -1 \rangle}$ is the *compositional inverse* of f if $f\left(f^{\langle -1 \rangle}(z)\right) = f^{\langle -1 \rangle}(f(z)) = z$.

Lemma 8.1. *A formal power series $f \in \mathbb{Q}[[z]]$ has a compositional inverse if and only if $f(0) = 0$ and $(Df)(0) \neq 0$.*

Suppose that $f(z)$ is defined implicitly by an equation of the form $f = z\phi(f)$ for some power series $\phi \in \mathbb{Q}[[z]]$ with $\phi(0) \neq 0$. Then, defining $g(z) = \dfrac{z}{\phi(z)}$, we have

$$g(f(z)) = \frac{f(z)}{\phi(f(z))} = z$$

showing that $z \mapsto \dfrac{z}{\phi(z)}$ is the compositional inverse of f.

The technique of Lagrange Inversion allows us to extract coefficients from f by considering coefficients of the compositional inverse of f.

Theorem 8.2 (Lagrange Inversion)**.** *Let $f \in \mathbb{Q}[[z]]$ with $f(0) = 0$ and $(Df)(0) \neq 0$. Suppose that f is defined implicitly by a functional equation $f = z\phi(f)$ for a formal power series $\phi \in \mathbb{Q}[[z]]$ with $\phi(0) \neq 0$. Then,*

$$[z^n]f(z) = \frac{1}{n}[z^{n-1}]\phi(z)^n.$$

There are many proofs of the theorem, some analytic and some algebraic. See [STANLEY, R. P. *Enumerative combinatorics. Vol. 2*, vol. 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999], Chapter 5, which gives three proofs.

A slight generalization of Lagrange Inversion will prove useful.

Theorem 8.3 (Lagrange Inversion)**.** *Let $f \in \mathbb{Q}[[z]]$ with $f(0) = 0$ and $(Df)(0) \neq 0$. Suppose that f is defined implicitly by a functional equation $f = z\phi(f)$ for a formal power series $\phi \in \mathbb{Q}[[z]]$ with $\phi(0) \neq 0$. Then,*

$$[z^n]f(z)^k = \frac{k}{n}[z^{n-1}]\phi(z)^n.$$

Let us now revisit some of our earlier results to see what information can be derived with this new tool at our disposal.

Example: Let $\mathcal{T}^{\{0,1,2\}}$ be the class of rooted planar unary-binary trees, i.e., every node has either 0, 1, or 2 children. Let the size of a tree be the total number of nodes. Then,

$$\mathcal{T}^{\{0,1,2\}} = \mathcal{Z} \times \textsc{Seq}_{\{0,1,2\}}(\mathcal{T}^{\{0,1,2\}}).$$

From this we find the functional equation

$$T(z) = z(1 + T(z) + T(z)^2).$$

Rearranging,

$$z = \frac{T}{\phi(T)},$$

for $\phi(z) = 1 + z + z^2$. Thus we can apply Lagrange inversion to conclude that

$$\begin{aligned}[z^n]T(z) &= \frac{1}{n}[z^{n-1}](1 + z + z^2)^n \\ &= \frac{1}{n}\sum_{k=1}^{\lfloor (n-1)/2 \rfloor} \binom{n}{k}\binom{n-k}{k+1}.\end{aligned}$$

This last step involves a simple combinatorial argument about how different combinations of powers of z can be found in terms of $(1 + z + z^2)^n$.

We will see more examples once we begin to cover labelled structures.