LECTURE 6 – ADDITIONAL CONSTRUCTIONS

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The constructions from the previous section ($+$, $\times$, SEQ) only get us so far. To build more complicated combinatorial classes, we need a counterpart to SEQ that builds *unordered* sets of objects from $\mathcal{A}$.

To that end, we define $\text{PSET}(\mathcal{A})$ to be the class of finite sets of objects from $\mathcal{A}$ (without repetition), where the size of a set is defined to be the sum of the sizes of the objects in the set. Similarly, define $\text{MSET}(\mathcal{A})$ to be to class of finite multisets of objects from $\mathcal{A}$, with the size again defined to be the sum of the sizes of the objects in the multiset.

For example, let $\mathcal{B}$ be the set of binary words. Then $\{00, 100, 1, 1111\}$ is in $\text{PSET}(\mathcal{B})$ and has size 10, while $\{1, 1, 00, 00, 1000\}$ is in $\text{MSET}(\mathcal{B})$ but not$\text{PSET}(\mathcal{B})$, and also has size 10.

Let $\mathcal{A}$ be an arbitrary class and let $\mathcal{C} = \text{PSET}(\mathcal{A})$. Assume for the moment that $\mathcal{A}$ is finite. Every set S can be built by choosing, for each $\alpha \in \mathcal{A}$, whether α should be included as part of S. If it is, it contributes $|\alpha|$ to the total size of S. Otherwise it contributes zero. Therefore[1],

$$\text{PSET}(\mathcal{A}) = \prod_{\alpha \in \mathcal{A}} (\{\epsilon\} + \{\alpha\}),$$

in much the same way that, for regular multiplication,

$$(1+a)(1+b)(1+c) = 1 + (a+b+c) + (ab+ac+bc) + abc.$$

Therefore, we derive the generating function

$$C(z) = \prod_{\alpha \in \mathcal{A}} (1 + z^{|\alpha|}) = \prod_{n=0}^{\infty} (1+z^n)^{a_n}.$$

It is often convenient to apply the exp–log transform.

$$\begin{aligned}
\prod_{n=0}^{\infty} (1+z^n)^{a_n} &= \exp\left(\log\left(\prod_{n=0}^{\infty}(1+z^n)^{a_n}\right)\right) \\
&= \exp\left(\sum_{n=0}^{\infty}\left(a_n \sum_{k=1}^{\infty} (-1)^{k-1}\frac{z^{nk}}{k}\right)\right) \\
&= \exp\left(\sum_{k=1}^{\infty}\left(\frac{(-1)^{k-1}}{k}\sum_{n=0}^{\infty} a_n z^{nk}\right)\right)
\end{aligned}$$

[1]The product symbol $\prod$ should be interpreted to mean an iterated Cartesian product construction.

$$= \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}A(z^k)\right)$$
$$= \exp\left(A(z) - \frac{A(z^2)}{2} + \frac{A(z^3)}{3} - \cdots\right).$$

The second equality used the identity

$$\log(1+u) = u - \frac{u^2}{2} + \frac{u^3}{3} - \cdots.$$

The third equality results from interchanging the summation signs, which we can do because we assumed that $\mathcal{A}$ was finite and thus the summation from $n = 1$ to ∞ is actually a finite sum.

It is easy to extend to the case where $\mathcal{A}$ is infinite. Suppose $\mathcal{A}$ is infinite and that $\mathcal{C} = \text{PSET}(\mathcal{A})$. Let $\mathcal{A}^{(\leq n)}$ be the class of elements in $\mathcal{A}$ of size at most n, and define $\mathcal{C}^{(\leq n)} = \text{PSET}(\mathcal{A}^{(\leq n)})$. From the generating function above we see that since $A(z)$ and $A^{(\leq n)}(z)$ agree up to z^n, and thus $C(z)$ and $C^{(\leq n)}(z)$ agree up to z^n. Combinatorially, this argument says that the elements of $\mathcal{C}$ of size at most n depend only on the (finitely many) elements of $\mathcal{A}$ of size at most n.

Taking the limit[2] as $n \to \infty$, we can conclude that the expression for $C(z)$ above is valid.

As it turns out, the MSET construction is usually more useful than the PSET construction. Let $\mathcal{C} = \text{MSET}(\mathcal{A})$. Unlike the PSET construction, for each $\alpha \in \mathcal{A}$, we can choose not only 0 or 1 occurrences of α, but also 2, 3, etc. Hence,

$$\text{MSET}(\mathcal{A}) = \prod_{\alpha\in\mathcal{A}} \text{SEQ}(\{\alpha\}).$$

Even though the SEQ construction forms ordered sequences, this is irrelevant because each sequence contains only one distinct element. Additionally, we see from this formula that we must stipulate that $\mathcal{A}$ has no neutral elements. Now, we see

$$C(z) = \prod_{\alpha\in\mathcal{A}}\left(\frac{1}{1-z^{|\alpha|}}\right) = \prod_{n=1}^{\infty}\left(\frac{1}{1-z^n}\right)^{a_n} = \prod_{n=1}^{\infty}(1-z^n)^{-a_n}.$$

We can simplify the infinite product using the *exp–log* transformation (and technically, by assuming first that $\mathcal{A}$ is finite and using a similar argument to the PSET case).

$$\prod_{n=1}^{\infty}(1-z^n)^{-a_n} = \exp\left(A(z) + \frac{A(z^2)}{2} + \frac{A(z^3)}{3} + \cdots\right).$$

Let's do some examples before moving on to the remaining construction.

Definition: A *partition* of n is an unordered sum of positive integers that sums to n. Thus $1+1+3$ and $3+1+1$ are the same partition of 5.

[2]We have not defined the ultrametric on formal power series, but two series are "close" when many of their initial coefficients agree.

Example: Let $\mathcal{P}$ be the class of partitions, with the size of the partitions defined to be the sum. Using the multiset construction, we see

$$\mathcal{P} = \text{MSET}(\mathcal{I}_{\geq 1}),$$

yielding the generating function

$$P(z) = \exp\left(I_{\geq 1}(z) + \frac{I_{\geq 1}(z^2)}{2} + \frac{I_{\geq 1}(z^3)}{3} + \cdots \right).$$

or, alternatively,

$$P(z) = \prod_{n=1}^{\infty} \frac{1}{1-z^n}.$$

Unlike the nice form $c_n = 2^{n-1}$ of compositions, there is no closed-form expression for the partition number p_n. Famous work of Hardy and Ramanujan proved that

$$p_n \sim \frac{1}{4n\sqrt{3}} \exp\left(\pi \sqrt{\frac{2n}{3}} \right).$$

Example: We can apply many of the same restrictions to partitions as compositions. We'll do only one example here. How many ways are there to make change for a dollar using pennies, nickels, dimes, and quarters? This is equivalent to asking for the number of partitions of 100 where the parts are restricted to the sizes 1, 5, 10, and 25. The class of such partitions is

$$\mathcal{M} = \text{MSET}(\{p, n, d, q\}),$$

where the sizes of the objects p, n, d, and q are 1, 5, 10, and 25, respectively. Therefore,

$$M(z) = \frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{25})}.$$

From this we can use Maple to find that $[z^{100}]M(z) = 242$.

Multisets also arise naturally in the consideration of nonplanar trees.

Example: Let $\mathcal{H}$ be the class of all rooted nonplanar trees (i.e., the order of the children is not significant). Let the size of a tree be the total number of nodes. Then,

$$\mathcal{H} = \mathcal{Z} \times \text{MSET}(\mathcal{H}).$$

Therefore, we have the functional equation

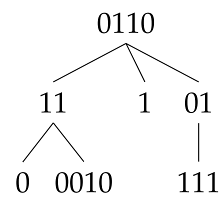
$$H(z) = z \exp\left(H(z) + \frac{H(z^2)}{2} + \frac{H(z^3)}{3} + \cdots \right).$$

While this has no closed form, it does allow for very fast computation of the coefficients H_n.

The book has a construction for *cycles* of objects. These are just sequences arranged in a circle so that translation of the sequence does not result in a different object. For example, if we consider cyclic binary words, then 111000 is considered the same word as 001110.

The construction is a bit involved, and not very useful in this chapter, so we're skipping it. We're skipping the *pointing* construction, as well.

The last construction we'll consider for now is SUBSTITUTION. The class SUBSTITUTION$(\mathcal{A}, \mathcal{B})$, more succinctly denoted $\mathcal{A}[\mathcal{B}]$, consists of all objects of $\mathcal{A}$, where each "component" of the size of $\mathcal{A}$ can be "inflated" be an entry of $\mathcal{B}$. As an example, let $\mathcal{A}$ be the class of rooted plane trees where all nodes contribute to size, and let $\mathcal{B}$ be the class of non-empty binary words. Then one element of $\mathcal{A}[\mathcal{B}]$ is

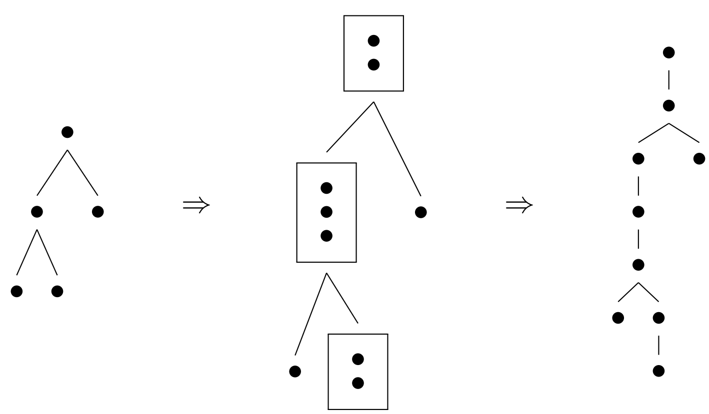


and this element has size 17. In order for $\mathcal{A}[\mathcal{B}]$ to be a combinatorial class, it is necessary that $\mathcal{B}$ has no elements of size 0. Otherwise, every component of every element of $\mathcal{A}$ could be inflated by element of size 0, giving an infinite number of size-0 objects in $\mathcal{A}[\mathcal{B}]$.

Let $\mathcal{C} = \mathcal{A}[\mathcal{B}]$. By the description given above, we see that

$$C(z) = \sum_{n=0}^{\infty} A_n B(z)^n = (A \circ B)(z).$$

Example: Let $\mathcal{M}$ be the set of all rooted planar unary-binary trees (all vertices have 0, 1, or 2 children). Every such tree can be formed by starting with a binary tree and turning every vertex into a vertical chain of vertices of length at least one.[3] See the picture below for an example.



Therefore

$$\mathcal{M} = \mathcal{B}[\text{SEQ}_{\geq 1}(\mathcal{Z}_\bullet)].$$

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[3]Perhaps the reverse operation is more obvious: every unary tree can be turned into a binary tree by contracting all chains of vertices with only 1 children.

We hence derive

$$M(z) = B\left(\frac{z}{1-z}\right) = \frac{1-\sqrt{1-4\left(\frac{z}{1-z}\right)^2}}{2\left(\frac{z}{1-z}\right)} = \frac{1-z-\sqrt{1-2z-3z^2}}{2z}.$$

The sequence of coefficients of $M(z)$ is known as the *Motzkin numbers*.

Example: A *balanced* 2-3 *tree* is a rooted planar tree whose internal nodes have degree 2 or 3 and all leaves are the same distance to the root. This tree has applications to computer science in the form of fast data storage. Let $\mathcal{T}$ be the class of balanced 2-3 trees where the size of a tree is the number of leaves. Then,

$$\mathcal{T} = \mathcal{Z} + \mathcal{T}[(\mathcal{Z} \times \mathcal{Z}) + (\mathcal{Z} \times \mathcal{Z} \times \mathcal{Z})],$$

and so the generating function satisfies

$$T(z) = z + T(z^2 + z^3).$$

We showed earlier how to extract coefficients of $T(z)$ quickly using Maple. Asymptotic expansion can be done, but it's very involved.

When we described the SEQ construction, we also detailed the admissible constructions for restricted versions, e.g., $\textsc{Seq}_{\leq k}$, $\textsc{Seq}_{\geq k}$, and $\textsc{Seq}_k$. For these examples, it was easy to calculate the associated generating functions.

Unfortunately, for PSET and MSET such easy translations are not available. This is due in large part to the complicated nature of these generating functions. We will be able to give full formulas for the restricted versions after we learn about bivariate generating functions. For now we must be content with the following results.

Theorem 6.1. *Let* $\mathcal{B} = \textsc{PSet}_2(\mathcal{A})$. *Then,*

$$B(z) = \frac{1}{2}A(z)^2 - \frac{1}{2}A(z^2).$$

Proof. The cartesian product of $\mathcal{A}$ with itself contains all pairs of distinct elements twice (once in each order), and all pairs of equal elements once. Define the diagonal construction by

$$\Delta(\mathcal{A}) = \{(\alpha, \alpha) : \alpha \in \mathcal{A}\},$$

with $|(\alpha, \alpha)| = 2|\alpha|$. If $\mathcal{C} = \Delta(\mathcal{A})$ then $C(z) = A(z^2)$.

Now, we can write

$$\mathcal{A} \times \mathcal{A} = \textsc{PSet}_2(\mathcal{A}) + \textsc{PSet}_2(\mathcal{A}) + \Delta(\mathcal{A}).$$

Thus,

$$A(z)^2 = 2B(z) + A(z^2),$$

and so

$$B(z) = \frac{1}{2}A(z)^2 - \frac{1}{2}A(z^2).$$

□

Exercise: Prove that if $\mathcal{B} = \text{MSET}_2(\mathcal{A})$ then $B(z) = \frac{1}{2}A(z)^2 + \frac{1}{2}A(z^2)$.

The generating function translation quickly becomes unmanageable.

Exercise: Prove that if $\mathcal{B} = \text{PSET}_3(\mathcal{A})$ then

$$B(z) = \frac{1}{6}A(z)^3 - \frac{1}{2}A(z)A(z^2) + \frac{1}{3}A(z^3).$$

Exercise: Prove that if $\mathcal{B} = \text{MSET}_3(\mathcal{A})$ then

$$B(z) = \frac{1}{6}A(z)^3 + \frac{1}{2}A(z)A(z^2) + \frac{1}{3}A(z^3).$$