LECTURE 4 – COMBINATORIAL CLASSES AND ADMISSIBLE CONSTRUCTIONS

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COMBINATORIAL CLASSES

We'll first formalize the notion of a set of combinatorial objects and their associated generating function.

Definition: A *combinatorial class* is a set $\mathcal{A}$ together with a size function $|\cdot|_{\mathcal{A}}$, such that

(1) the size of an element is a non-negative integer, and

(2) the number of elements of a given size is finite.

For a class $\mathcal{A}$, we define $\mathcal{A}_n$ to be the objects in $\mathcal{A}$ of size n and a_n to be the number of objects in $\mathcal{A}$ of size n. The *counting sequence* of a class is the sequence $(a_n)_{n\geq 0}$, and the generating function of a class is

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = \sum_{\alpha\in\mathcal{A}} z^{|\alpha|}.$$

Think of the generating function as a collapsing of the class, where all internal structure other than size is ignored. This is analogous to describing the chemical molecule glucose as $C_6H_{12}O_6$ rather than by its bond diagram. (Other sugars also have the $C_6H_{12}O_6$ form, though with different bond diagrams.)

Example: Let $\mathcal{B}$ be the class of binary words over the alphabet $\{0,1\}$. Define $|\beta|$ to be the length of β. The elements of $\mathcal{B}$ are

$$\mathcal{B} = \{\epsilon, 0, 1, 00, 01, 10, 11, 000, \ldots\},$$

where ϵ is the "empty word" of length zero.

We already know that $a_n = 2^n$, and it follows that $B(z) = \dfrac{1}{1-2z}$, but the symbolic method is all about going the other way around. We'll find the generating function *directly from the specification of the class*, then use it to find the closed form of b_n (if possible).

ADMISSIBLE CONSTRUCTIONS

The symbolic method allows us to *construct* new combinatorial classes from old ones. A construction is a function Φ that inputs a finite sequence of classes and outputs a class:

$$\mathcal{C} = \Phi[\mathcal{A}_1, \ldots, \mathcal{A}_k].$$

A construction is *admissible* if the generating function of the resulting class can be derived from only the generating functions of the input classes (i.e., without any additional structural information about the input classes). So, for each admissible construction $\Phi : (\mathcal{A}_1, \dots \mathcal{A}_k) \mapsto \mathcal{C}$, there is a function ϕ such that

$$C(z) = \phi[A_1(z), \dots, A_k(z)].$$

Example: Define $\textsc{OrdPairs}(\mathcal{A})$ to be the class of objects that are ordered pairs of objects from $\mathcal{A}$. Formally,

$$\mathcal{C} = \textsc{OrdPairs}(\mathcal{A}) = \{(\alpha_1, \alpha_2) : \alpha_1, \alpha_2 \in \mathcal{A}\},$$

with size function

$$|(\alpha_1, \alpha_2)|_{\mathcal{C}} = |\alpha_1|_{\mathcal{A}} + |\alpha_2|_{\mathcal{A}}.$$

As a quick example, if $\mathcal{A}$ is the class of binary words over $\{0, 1\}$, then three distinct elements of $\textsc{OrdPairs}(\mathcal{A})$ are $(001, \epsilon)$, $(0, 01)$, and $(01, 0)$, and all have size 3.

An object of size n in $\mathcal{C} = \textsc{OrdPairs}(\mathcal{A})$ consists of an object of size k from $\mathcal{A}$ and an object of size $n - k$ from $\mathcal{A}$, for all $0 \leq k \leq n$ (and order matters). Therefore,

$$\begin{aligned}
[z^n]C(z) &= \sum_{k=0}^{n} \Big([z^k]A(z)\Big) \cdot \Big([z^{n-k}]A(z)\Big) \\
&= \sum_{k=0}^{n} a_k a_{n-k} \\
&= [z^n]A(z)^2
\end{aligned}$$

Therefore,

$$C(z) = A(z)^2.$$

(As a general rule, the product of generating functions tells you the number of ordered ways to "pick one thing" from each generating function, where sizes are additive.)

Since we'll always have $C(z) = A(z)^2$ regardless of the structure of $\mathcal{A}$, this construction is admissible. We can now apply it to any combinatorial class. Suppose we want to find the generating function for ordered pairs of binary words. Let $\mathcal{B}$ be the class of binary words (as above), so that $B(z) = \dfrac{1}{1 - 2z}$. Then, the class of ordered pairs of binary words is

$$\mathcal{C} = \textsc{OrdPairs}(\mathcal{B}).$$

Therefore, we obtain immediately that

$$C(z) = \left(\frac{1}{1 - 2z}\right)^2 = \frac{1}{(1 - 2z)^2},$$

and so $c_n = (n + 1)2^n$. Of course, this is easy to figure out directly. The real constructions we build will be less trivial.

Non-Example: Given two classes $\mathcal{A}$ and $\mathcal{B}$, define $\mathcal{C} = \textsc{Union}(\mathcal{A}, \mathcal{B})$ by

$$\mathcal{C} = \{\gamma : \gamma \in \mathcal{A} \text{ or } \gamma \in \mathcal{B}\}.$$

One potential problem arises if $\gamma \in \mathcal{A} \cap \mathcal{B}$ but $|\gamma|_{\mathcal{A}} \neq |\gamma|_{\mathcal{B}}$. Let's assume that this doesn't happen, i.e., that the size function is consistent across $\mathcal{A}$ and $\mathcal{B}$.

If this construction is to be admissible, then we must be able to calculate $C(z)$ from $A(z)$ and $B(z)$ directly. This is a problem when $\mathcal{A} \cap \mathcal{B} \neq \emptyset$.

Consider for example the case where $\mathcal{A}$ is the class of all words over the alphabet $\{0, 1\}$, $\mathcal{B}$ is the class of all words over the alphabet $\{1, 2\}$, and $\mathcal{C}$ is the class of all words over the alphabet $\{7, 8\}$. Clearly, the number of words of length n in $\textsc{Union}(\mathcal{A}, \mathcal{B})$ is $2^{n+1} - 1$, while the number of words of length n in $\textsc{Union}(\mathcal{A}, \mathcal{C})$ is 2^{n+1}.

Therefore, for arbitrary classes $\mathcal{A}$ and $\mathcal{B}$, we cannot compute the generating function of $\textsc{Union}(\mathcal{A}, \mathcal{B})$, and so UNION is not an admissible construction.

$\mathcal{E}$, $\mathcal{Z}$, DISUNION, CARTPROD, AND SEQ

Elemental Classes. As the symbolic method lets us build new classes from old, we need some simple classes to be the initial building blocks.

Let $\mathcal{E}$ be the class that consists of a single element that has size 0. We call this element ϵ and call it the *neutral object*. The generating function of $\mathcal{E}$ is $E(z) = 1$.

Let $\mathcal{Z}$ be the class that consists of a single element that has size 1. We will often subscript the class name for clarity. For example, the classes that represent the single word a of length 1 and the single word b of length 1 can be denoted $\mathcal{Z}_\mathsf{a}$ and $\mathcal{Z}_\mathsf{b}$, respectively. We call these *atomic elements*. The generating function for $\mathcal{Z}$ is $Z(z) = z$.

DISUNION. The construction UNION discussed earlier was shown to not be an admissible construction. We consider instead the DISUNION construction in which $\mathcal{A}$ and $\mathcal{B}$ are assumed to be (well, are forced to be) disjoint.

To disjointify $\mathcal{A}$ and $\mathcal{B}$, we imagine that all elements of $\mathcal{A}$ are colored **red** and all elements of $\mathcal{B}$ are colored **blue**. With this convention, the construction becomes admissible. Let $\mathcal{C} = \textsc{DisUnion}(\mathcal{A}, \mathcal{B})$. Then

$$[z^n]C(z) = [z^n]A(z) + [z^n]B(z),$$

and therefore $C(z) = A(z) + B(z)$. Of course, DISUNION performs a different job than UNION, so if your input classes $\mathcal{A}$ and $\mathcal{B}$ are not really disjoint, make sure the disjointification is what you want.

We abbreviate $\textsc{DisUnion}(\mathcal{A}, \mathcal{B}) = \mathcal{A} + \mathcal{B}$.

CARTPROD. The ORDPAIRS construction above allowed us to form all ordered pairs of objects from a class $\mathcal{A}$. The CARTPROD construction is more general: it forms all ordered pairs of an object from $\mathcal{A}$ (first) and an object from $\mathcal{B}$ (second). Note that

$$\textsc{OrdPairs}(\mathcal{A}) = \textsc{CartProd}(\mathcal{A}, \mathcal{A}).$$

The size function is additive: if $\alpha \in \mathcal{A}$ and $\beta \in \mathcal{B}$, then $|(\alpha, \beta)| = |\alpha|_\mathcal{A} + |\beta|_\mathcal{B}$.

Because the pairs are ordered, we don't need to worry about disjointness of $\mathcal{A}$ and $\mathcal{B}$. However, beware that you *cannot* form all *unordered* pairs using the construction

$$\textsc{CartProd}(\mathcal{A}, \mathcal{B}) + \textsc{CartProd}(\mathcal{B}, \mathcal{A})$$

if $\mathcal{A} \cap \mathcal{B} \neq \emptyset$.

Let $\mathcal{C} = \textsc{CartProd}(\mathcal{A}, \mathcal{B})$. The calculation of $C(z)$ is nearly identical to the $\textsc{OrdPairs}$ construction. The elements of size n in $\mathcal{C}$ are those with a first component of size k and a second component of size $n-k$, for all $0 \leq k \leq n$. Therefore,

$$\begin{aligned}[z^n]C(z) &= \sum_{k=0}^{n} \Big([z^k]A(z)\Big) \cdot \Big([z^{n-k}]B(z)\Big) \\ &= \sum_{k=0}^{n} a_k b_{n-k} \\ &= [z^n]A(z)B(z),\end{aligned}$$

proving that $C(z) = A(z)B(z)$.

We abbreviate $\textsc{CartProd}(\mathcal{A}, \mathcal{B}) = \mathcal{A} \times \mathcal{B}$.

By iteration, we can define the n-fold Cartesian product analogously: define

$$\mathcal{C} = \mathcal{A}_1 \times \mathcal{A}_2 \times \mathcal{A}_3 \times \cdots \mathcal{A}_k = (((\mathcal{A}_1 \times \mathcal{A}_2) \times \mathcal{A}_3) \times \cdots \times \mathcal{A}_k).$$

Then,

$$C(z) = A_1(z)A_2(z)A_3(z)\cdots A_k(z).$$

$\textsc{Seq}$. The last construction that we define for now is the sequence constructor. Given a class $\mathcal{A}$, define $\textsc{Seq}(\mathcal{A})$ to be the set of all finite (possibly zero) length sequences of elements from $\mathcal{A}$, with additive size. Formally,

$$\textsc{Seq}(\mathcal{A}) = \{\epsilon\} \cup \{(\alpha_1, \alpha_2, \ldots, \alpha_k) : k \in \mathbb{N}^+,\ \alpha_i \in \mathcal{A}\},$$

with

$$|(\alpha_1, \alpha_2, \ldots, \alpha_k)|_{\textsc{Seq}(\mathcal{A})} = |\alpha_1|_{\mathcal{A}} + |\alpha_2|_{\mathcal{A}} + \cdots + |\alpha_k|_{\mathcal{A}}.$$

Actually, this isn't really a new construction, because we can build it from $+$ and $\times$.

$$\textsc{Seq}(\mathcal{A}) = \underbrace{\mathcal{E}}_{\substack{\text{empty}\\ \text{seq}}} + \underbrace{\mathcal{A}}_{\substack{\text{seqs of}\\ \text{length 1}}} + \underbrace{\mathcal{A} \times \mathcal{A}}_{\substack{\text{seqs of}\\ \text{length 2}}} + \underbrace{\mathcal{A} \times \mathcal{A} \times \mathcal{A}}_{\substack{\text{seqs of}\\ \text{length 3}}} + \cdots.$$

(We're using the fact that the unions are all disjoint!)

From this we see that if $\mathcal{C} = \textsc{Seq}(\mathcal{A})$, then

$$\begin{aligned}C(z) &= 1 + A(z) + A(z)^2 + A(z)^3 + \cdots \\ &= \frac{1}{1 - A(z)}.\end{aligned}$$

This last line should concern you! We've taken an infinite sum of formal power series and declared that the result is also a formal power series. In fact, without a restriction on $A(z)$, this isn't even true.

If $\mathcal{A}$ has an elements of size zero, then each term in the sum forming $C(z)$ has a constant term of at least 1, so the result cannot be a formal power series. Suppose now that $\mathcal{A}$ has no element of size zero. Then, the smallest power of z with a nonzero coefficient in $A(z)^n$ is z^n. This tells us that the coefficient of z^n in $C(z)$ depends only on finitely many terms in the infinite sum on the right-hand side, and you can check that that coefficient is equal to the one in $(1 - A(z))^{-1}$, so we don't really need a formalized notion of infinite convergence of formal power series[1]. This argument looks familiar because it is. Since

$$\frac{1}{1 - A(z)} = \left(\frac{1}{1-z}\right) \circ A(z),$$

we actually already knew that as long as $a_0 = 0$,

$$\begin{aligned} 1 + A(z) + A(z)^2 + \cdots &= \left(1 + z + z^2 + \cdots\right) \circ A(z) \\ &= \left(\frac{1}{1-z}\right) \circ A(z) \\ &= \frac{1}{1 - A(z)}. \end{aligned}$$

Restricted Sequences

There are a number of modifications to the SEQ construction that will prove to be useful. As defined above, $\textsc{Seq}(\mathcal{A})$ consists of all sequences of elements from $\mathcal{A}$ of any length including zero. Often, we will want to restrict to only sequences of certain lengths. For example,

$$\begin{aligned} \textsc{Seq}_{=k}(\mathcal{A}) : &\quad \text{sequences of length exactly } k, \\ \textsc{Seq}_{\le k}(\mathcal{A}) : &\quad \text{sequences of length at most } k, \\ \textsc{Seq}_{\ge k}(\mathcal{A}) : &\quad \text{sequences of length at least } k, \\ \textsc{Seq}_{\text{even}}(\mathcal{A}) : &\quad \text{sequences of even length}, \\ \textsc{Seq}_{\text{odd}}(\mathcal{A}) : &\quad \text{sequences of odd length}, \end{aligned}$$

etc. Their generating functions are easily found:

$$\begin{aligned} \textsc{Seq}_{=k}(\mathcal{A}) : &\quad A(z)^k, \\ \textsc{Seq}_{\le k}(\mathcal{A}) : &\quad 1 + A(z) + A(z)^2 + \cdots + A(z)^k = \frac{1 - A(z)^{k+1}}{1 - A(z)}, \\ \textsc{Seq}_{\ge k}(\mathcal{A}) : &\quad A(z)^k + A(z)^{k+1} + \cdots = \frac{A(z)^k}{1 - A(z)}, \\ \textsc{Seq}_{\text{even}}(\mathcal{A}) : &\quad 1 + A(z)^2 + A(z)^4 + \cdots = \frac{1}{1 - A(z)^2}, \\ \textsc{Seq}_{\text{odd}}(\mathcal{A}) : &\quad A(z) + A(z)^3 + A(z)^5 + \cdots = \frac{A(z)}{1 - A(z)^2}. \end{aligned}$$

More generally, for any subset $\Omega \subseteq \mathbb{N}$, we define

$$\textsc{Seq}_{\Omega}(\mathcal{A}) = \sum_{\omega \in \Omega} \textsc{Seq}_{\omega}(\mathcal{A}),$$

[1]Such a notion exists, and can be quite useful. Let $f \in R[[z]]$, and define the valuation of f (denoted $\text{val}(f)$) to be the smallest-order coefficient that is nonzero. Define $d(f, g) = 2^{-\text{val}(f-g)}$, so that pairs of sequences with many of their first coefficients equal are close together. This is distance function forms a metric space (in fact, an ultrametric space).

where the summation sign should be interpreted as repeated application of the disjoint union construction. The associated generating function is

$$C(z) = \sum_{\omega \in \Omega} A(z)^{\omega}.$$