LECTURE 2 – GENERATING FUNCTIONS AND FORMAL POWER SERIES

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The Ring of Formal Power Series

Let R be a commutative ring. The *ring of formal power series* $R[[z]]$ is the set of all "infinite polynomials"

$$a_0 + a_1 z + a_2 z^2 + \cdots = \sum_{n=0}^{\infty} a_n z^n,$$

with $a_n \in R$. Typically, we'll use $R = \mathbb{Q}, \mathbb{R}, \mathbb{C}$.

The indeterminate z is just a placeholder. The elements of $R[[z]]$ are really just infinite sequences $(a_0, a_1, a_2, \ldots)$. We use the "infinite polynomial" notation because it's convenient.

Addition and multiplication in the ring are generalizations of those for polynomials:

- $\left(\sum_{n=0}^{\infty} a_n z^n\right) + \left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty} (a_n + b_n) z^n$
- $(a_0 + a_1 z + a_2 z^2 + \cdots)(b_0 + b_1 z + b_2 z^2 + \cdots)$
  $= a_0 b_0 + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + \cdots$

  $$\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{n=0}^{\infty} b_n z^n\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} a_k b_{n-k}\right) z^n$$

  This is called the *Cauchy product*.

The additive identity is 0 and the multiplicative identity is 1. There's a lot of algebraic theory on these rings that we don't need right now.

We don't care about convergence. For example, $\sum_{n=0}^{\infty} n! z^n$ is a perfectly valid formal power series.

Given $f = \sum_{n=0}^{\infty} a_n z^n \in R[[z]]$, define the *formal derivative* of f, denoted Df to be the formal power series

$$Df = \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n.$$

The differentiation operator works exactly as expected:

- $D(af + bg) = a(Df) + b(Dg)$, for $a, b \in R$ and $f, g \in \mathbb{R}[[z]]$.
- $D(fg) = f(Dg) + (Df)g$, for $f, g \in \mathbb{R}[[z]]$.

A version of the chain rule holds as well, but only if we're careful about composition.

Problem: Let $f = \sum_{n=0}^{\infty} z^n$. What is $f \circ f$?

$$\begin{aligned} f \circ f &= \sum_{n=0}^{\infty} f^n \\ &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} z^k \right)^n . \end{aligned}$$

This is *not* a formal power series. (Hint: Examine the constant term.)

Definition: Let $f, g \in R[[z]]$ and assume that the constant term of g is zero. Then, $f \circ g$ is a formal power series defined by

$$f \circ g = \sum_{n=0}^{\infty} f_n g^n.$$

This solves the problem because only finitely many terms of the sum contribute to each term of $f \circ g$. (Composition can be valid in other contexts as well, e.g., if only finitely many coefficients of f are nonzero.)

The coefficients of the composition are not too hard to compute. For $n \geq 1$, the coefficient of z^n in $f \circ g$ is

$$\sum_{(k,c) \in S} f_k g_{c_1} g_{c_2} \cdots g_{c_k},$$

where S is the set of all pairs (k, c) such that k is positive integer and c is a composition of n into k parts.[1]

We can now state the chain rule:

$$D(f \circ g) = (D(f) \circ g) D(g).$$

(Ordinary) Generating Functions

Note that $f(0) = a_0$, $(Df)(0) = a_1$, $(D^2 f)(0) = 2a_2$, and in general $(D^k f)(0) = k! a_k$. This is the link between formal power series and Taylor series of functions.

When the formal power series $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is equal to the Taylor series of a function $g(z)$ at $z = 0$, we consider them equal.

Examples:

- $\dfrac{1}{1-z} = 1 + z + z^2 + \cdots = \sum_{n=0}^{\infty} z^n, \qquad a_n = 1$

[1] This assumes $g_0 = 0$. If operating under a different assumption that makes composition valid, then weak compositions should be used.

- $\dfrac{1}{1-rz} = 1 + rz + r^2z^2 + \cdots = \displaystyle\sum_{n=0}^{\infty} r^n z^n, \qquad a_n = r^n$

- $\dfrac{1}{(1-z)^2} = 1 + 2z + 3z^2 + \cdots = \displaystyle\sum_{n=0} (n+1)z^n, \qquad a_n = n+1$

- $\dfrac{1}{(1-rz)^k} = 1 + krz + \dbinom{k+1}{2} r^2z^2 + \cdots = \displaystyle\sum_{n=0} \binom{n+k-1}{k-1} r^n z^n$

- $\log\left(\dfrac{1}{1-z}\right) = z + \dfrac{1}{2}z^2 + \dfrac{1}{3}z^3 + \cdots = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n} z^n$

- $\exp(z) = 1 + z + \dfrac{1}{2}z^2 + \dfrac{1}{6}z^3 + \cdots = \displaystyle\sum_{n=0}^{\infty} \frac{1}{n!} z^n$

Definition: Let $f(z) = \displaystyle\sum_{n=0}^{\infty} a_n z^n$. Then $[z^n]f(z) = a_n$.

Example: $[z^3]\dfrac{5}{(1-z)^2} = 20.$

Example: $[z^n]\dfrac{1-\sqrt{1-4z}}{2z} = \dfrac{1}{n+1}\dbinom{2n}{n}$. (We'll see how to do this later.)

You can use a CAS to get series expansions of generating functions.

Maple

```
> f := 5/(1-z)^2:
> series(f,z,10);
                   2        3        4        5        6        7        8        9        10
  5 + 10 z + 15 z  + 20 z  + 25 z  + 30 z  + 35 z  + 40 z  + 45 z  + 50 z  + O(z  )

> coeff(%,z,3);
                                        20
```

Sage

```
sage: var('z');
sage: f = 5/(1-z)^2;
sage: f.series(z,8)
         5 + 10*z + 15*z^2 + 20*z^3 + 25*z^4 + 30*z^5 + 35*z^6 + 40*z^7 + Order(z^8)
sage: _.coefficient(z,3)
         20
```

Transformations. In future lectures, we'll explore composition of generating functions in more detail, but for now it suffices to see that we can make simple substitutions to construct new generating functions.

Examples:

- $\dfrac{1}{1+z} = \dfrac{1}{1-(-z)} = \sum_{n=0}^{\infty}(-z)^n = \sum_{n=0}^{\infty}(-1)^n z^n = 1 - z + z^2 - z^4 + \cdots$

- $\dfrac{1}{1-z^3} = \sum_{n=0}^{\infty}(z^3)^n = 1 + z^3 + z^6 + \cdots$

- If $[z^n]f(z) = a_n$, then $[z^n]f(rz) = r^n a_n$

Other types of generating functions.

- **Multivariate OGF:** $\{a_{n,k}\}_{k,n\in\mathbb{N}} \longrightarrow \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_{n,k}x^n y^k$
- **Exponential GF:** $\{a_n\}_{n\in\mathbb{N}} \longrightarrow \sum_{n=0}^{\infty} a_n \dfrac{z^n}{n!}$
- Dirichlet, Bell, Lambert, ...