LECTURE 1 – WHAT IS AN ANSWER?

JAY PANTONE

Course Information

- **Office Hours:**
  - Monday, 4-5
  - Tuesday, 2-3
  - Thursday 10-11
- **Textbooks:**
  - *Analytic Combinatorics*, Flajolet and Sedgewick
  - *Generatingfunctionology*, Wilf
  - Various other sources
- **Assignments:** 5-6ish (TBD), a few questions each
- **Exams:** None
- No class Feb 15,17,19. We'll use three X-hours (Thursdays 1:00-1:50) on days there is no combo seminar.

Introduction to Enumerative Combinatorics

The goal of enumerative combinatorics is to *count things*. This can mean many different things.

Let $\mathcal{A}$ be a set of objects, and let a_n be the number of objects of size n in $\mathcal{A}$. There are at least four kinds of "answers" to the problem of finding a_n:

(1) **Closed form:** Some kind of formula to directly compute a_n for any given n. Possible examples include

$$a_n = 4^n + 3^n,$$
$$a_n = \sum_{k=0}^{n} (-1)^k \binom{n}{k}^2,$$
$$a_n = \left\lfloor \frac{2n+3}{5} \right\rfloor.$$

($\lfloor x \rfloor$ is the biggest integer smaller than x and $\lceil x \rceil$ is the smallest integer bigger than x)

We clearly want to avoid non-helpful formulas such as:

$$\sum_{\alpha \in \mathcal{A}_n} 1$$

where $\mathcal{A}_n$ is the set of objects in $\mathcal{A}$ of size n. In some sense, we want a closed formula that actually allows faster computation.

Suggested Reading: WILF, H. S. What is an answer? *Amer. Math. Monthly 89*, 5 (1982), 289–292

(2) **Recurrence:** Some formula to find a_n in terms of $a_0, a_1, \ldots, a_{n-1}$, along with initial terms. Examples:

$$\begin{aligned} a_n &= a_{n-1} + 2a_{n-3}, && a_0 = 1, a_1 = 1, a_2 = 5, \\ a_n &= n^2 a_{n-2}, && a_0 = 3, a_1 = 1, \\ a_n &= \sum_{k=0}^{n-1} a_k a_{n-k-1}, && a_0 = 2. \end{aligned}$$

(3) **Generating function:** The generating function of a sequence a_0, a_1, a_2 is the *formal sum*

$$f(z) = a_0 + a_1 z + a_2 z^2 + \cdots .$$

We don't care about convergence. Think of it like a data structure.

"A generating function is a clothesline on which we hang up a sequence of numbers for display." – Herb Wilf

At this point, a generating function is a purely algebraic object (in the ring of rational formal power series).

More on this later.

(4) **Asymptotic behavior:** Often, one is happy to just have an approximation of a_n for large n.

Definition: We say that $f_n \sim g_n$ if $\lim_{n\to\infty} \frac{f_n}{g_n} = 1$.

Examples:

$$\begin{aligned} a_n &= n^2 + n + 1 && \Longrightarrow && a_n \sim n^2, \\ a_n &= 4^n + 10^5 \cdot 3.99^n && \Longrightarrow && a_n \sim 4^n. \end{aligned}$$

Each type of answer has pros and cons.

Example: Let a_n be the number of ways to climb a flight of n stairs, taking the stairs either one or two at a time.

Closed form: Not obvious right away.

Recurrence: You can reach the nth step by taking a step of size 1 from the $(n-1)$th step or a step of size 2 from the $(n-2)$th step. Hence:

$$a_n = a_{n-1} + a_{n-2},$$

with initial conditions $a_1 = 1$ and $a_2 = 2$. These are called the *Fibonacci numbers*.

Generating function: The symbolic method will allow us to *immediately* derive the generating function

$$f(x) = \frac{1}{1-z-z^2}.$$

This means that the Taylor series expansion of $\frac{1}{1-z-z^2}$ gives the formal power series of the correct coefficients, i.e.,

$$\frac{1}{1-z-z^2} = a_0 + a_1 z + a_2 z^2 + \cdots .$$

The whole point is that we can find this generating function without knowing a closed form expression for a_n.

Asymptotic behavior: Not obvious right away.

In questions like this, often the generating function is the easiest thing to find. From the generating function, we can sometimes find a closed form ("coefficient extraction"), the recurrence, and asymptotic behavior.

In this example, we can find a closed form:

$$a_n = \frac{\sqrt{5}+5}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{\sqrt{5}-5}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

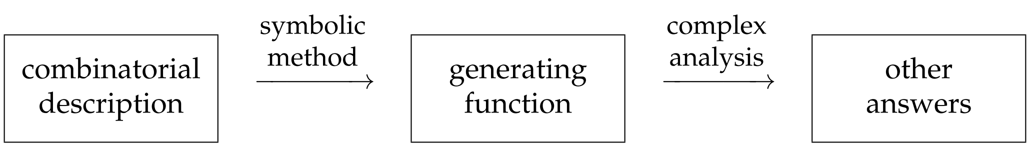
Note that $\frac{1+\sqrt{5}}{2} \approx 1.618\ldots$ and $\frac{1-\sqrt{5}}{2} \approx -0.618\ldots$. So, as $n \to \infty$,

$$a_n \sim \frac{\sqrt{5}+5}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

This estimation is very good:

n	a_n	$\frac{\sqrt{5}+5}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n$
0	1	0.723...
1	1	1.170...
2	2	1.894...
3	3	3.065...
4	5	4.959...
5	8	8.024...
6	13	12.984...
7	21	21.009...
8	34	33.994...
100	573147844013817084101	573147844013817084100.9999999999999999999996...

The guiding principal of this course is that the symbolic method may be used to obtain a generating function, and complex analysis may be used to derive information about the counting sequence.



The symbolic method will be covered in the first part of the course, and analytic combinatorics will be covered in the second.