MATH 118, WINTER '16

HOMEWORK 1

Due Wednesday, Jan 20

1. Let $f, g \in R[[z]]$ and let D be the differentiation operator on formal power series.
   a) Prove that $D(fg) = D(f)g + fD(g)$.
   b) Prove that $D(f^n) = nf^{n-1}D(f)$.
2. Let $f(z)$ be the generating function for the sequence $\{a_0, a_1, \ldots\}$. Prove that $\dfrac{f(z)}{1-z}$ is the generating function for the sequence of partial sums $\{a_0, a_0 + a_1, a_0 + a_1 + a_2, \ldots\}$.
3. Prove that the generating function $\exp(z)$ is not algebraic. Your proof should be on the level of formal power series. Do not use Eisenstein's theorem. (*Hint:* Suppose it is, write the minimal polynomial it satisfies, and find a contradiction.)
4. ~~Prove that D-algebraic generating functions have Wilfian formulas. It may help to try to prove the D-finite case first.~~ Prove that D-finite generating functions have Wilfian formulas.
5. Find a symbolic specification and then the generating function for words over the alphabet $\{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$ that do not contain k consecutive a's or ℓ consecutive b's. For imaginary extra credit, find the number of these words of length 30.
6. In this exercise we will use basic generating function techniques to prove that

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{12\sqrt{3} + 2\pi}{9\sqrt{3}}.$$

   (a) Let $f(z) = \displaystyle\sum_{n=0}^{\infty} \binom{2n}{n}^{-1} z^n$. Verify that $f(z)$ is D-finite by showing that $f(z)$ satisfies the linear differential equation[1]

$$0 = z(z-4)f'(z) + (z+2)f(z) - 2.$$

   (b) Using Mathematica[2], we find that the solution to the linear ODE with initial condition $f(0) = 1$ is

$$f(z) = \frac{4}{4-z} + \frac{4\sqrt{z}\arcsin\left(\frac{\sqrt{z}}{2}\right)}{(4-z)^{3/2}}.$$

   Use this to prove that

$$\sum_{n=0}^{\infty} \frac{1}{\binom{2n}{n}} = \frac{12\sqrt{3} + 2\pi}{9\sqrt{3}}.$$

   (*Note:* You have to say something about convergence.)

[1] Forming the conjecture that $f(z)$ satisfies this ODE is actually quite easy. There is a very simple algorithm that I hope to cover later in the course.

[2] Maple can't do this for some reason