MATH 31 – HOMEWORK 8

due Wednesday, August 23

Instructions: This assignment is due at the *beginning* of class. Staple your work together (do not just fold over the corner). Please write the questions in the correct order. If I cannot read your handwriting, you won't receive full credit.

1. Let R be a commutative ring with unity (i.e., with a 1) and suppose $a, b \in R$.
   (a) Prove that if ab is a unit, then both a and b are units.
   (b) Prove that if ab is a zero-divisor, then either a is a zero-divisor or b is a zero-divisor.
   (c) Suppose R is a domain. Prove that if $a, b \in R$ are such that $a^2 = b^2$, then $a = b$ or $a = -b$.
2. (16.7) Let X be a set and let $R = \mathcal{P}(X)$.
   (a) Show that $(R, \triangle, \cap)$ is a ring, where $\triangle$ denotes the symmetric difference and $\cap$ denotes the intersection.
   (b) A *Boolean ring* is one in which every element a has the property that $a^2 = a$. Show that $(R, \triangle, \cap)$ is a Boolean ring.
3. (17.7) Find all the maximal ideals in the ring $(\mathbb{Z}_n, \oplus, \otimes)$.
4. (17.10) Let X be a nonempty set and let R be the ring $(\mathcal{P}(X), \triangle, \cap)$.
   (a) Show that if $Y \subsetneq X$ (this symbol means that Y is a subset of X and $Y \neq X$), then $\mathcal{P}(Y)$ is an ideal in R and has a multiplicative identity different from that of R.
   (b) Find a maximal ideal in R.
5. (18.14) Let $\varphi : R \to S$ be a ring homomorphism and suppose that S has a multiplicative identity 1_S. Show that $\varphi^{-1}(\{1_S\})$ is an ideal in R if and only if S is trivial.