MATH 31 – HOMEWORK 6 SOLUTIONS!

due Wednesday, August 9

Instructions: This assignment is due at the *beginning* of class. Staple your work together (do not just fold over the corner). Please write the questions in the correct order. If I cannot read your handwriting, you won't receive full credit.

1. (12.13) Let $\varphi : G \to H$ be a homomorphism.

   (a) Show that if H is abelian and φ is one-to-one, then G is abelian.
   Proof: Suppose H is abelian and φ is one-to-one. Let $a, b \in G$ be arbitrary. We aim to show that $ab = ba$.
   By the rules of homomorphisms,
   $$\varphi(ab) = \varphi(a)\varphi(b)$$
   and
   $$\varphi(ba) = \varphi(b)\varphi(a).$$
   As H is abelian,
   $$\varphi(a)\varphi(b) = \varphi(b)\varphi(a)$$
   and so
   $$\varphi(ab) = \varphi(ba).$$
   The injectivity of φ implies that $ab = ba$. □

   (b) Show that if G is abelian and φ is onto, then H is abelian.
   Proof: Suppose that G is abelian and φ is onto. Let $x, y \in H$ be arbitrary. We aim to show that $xy = yx$.
   Since φ is onto, there exist $a, b \in G$ such that $\varphi(a) = x$ and $\varphi(b) = y$. Now,
   $$xy = \varphi(a)\varphi(b) = \varphi(ab)$$
   and
   $$yx = \varphi(b)\varphi(a) = \varphi(ba).$$
   As G is abelian, $ab = ba$. Therefore,
   $$xy = \varphi(ab) = \varphi(ba) = yx.$$
   Therefore G is abelian. □

   (c) Show that if φ is an isomorphism then G is abelian if and only if H is.
   Proof: Suppose φ is an isomorphism.
   If G is abelian, then part (b) shows that H is abelian because isomorphisms must be surjective. If H is abelian, there part (a) shows that G is abelian because isomorphisms must be injective. □

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2. (12.21) Let G be the group of nonzero complex numbers under multiplication and let H be the subgroup of $GL(2,\mathbb{R})$ consisting of all matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where not both a and b are 0. Show that $G \cong H$.

   Solutions: To prove that $G \cong H$, we will define a function and prove that it's an isomorphism.

   We need a function whose input is a nonzero complex number $a + bi$ and whose output is a nonzero matrix of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$. The question was phrased in such a way to hint at the map

$$\varphi(a+bi) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

   The map makes sense because we have to start with *nonzero* complex numbers, and therefore the outputs of φ really are matrices of the proper form, with a and b not both 0.

   First, we'll check that φ is a homomorphism:

$$\varphi((a+bi)(c+di)) = \varphi((ac-bd)+(ad+bc)i) = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix}$$

   while

$$\varphi(a+bi)\varphi(c+di) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac-bd & ad+bc \\ -(ad+bc) & ac-bd \end{pmatrix}.$$

   Next, we'll check that φ is surjective. Let $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ be a matrix where x and y are not both zero. Then, $\varphi(x+yi) = \begin{pmatrix} x & y \\ -y & x \end{pmatrix}$, and moreover $x + yi \neq 0$, so $x + yi$ really is in the domain.

   Lastly, we'll verify that φ is injective. Let $r + si$ and $u + vi$ be two nonzero complex numbers such that $\varphi(r + si) = \varphi(u + vi)$. Then,

$$\begin{pmatrix} r & s \\ -s & r \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix}.$$

   It follows that $r = u$ and $s = v$ so that $r + si = u + vi$. This confirms that φ is injective.

   Having proved that φ is a homomorphism, injective, and surjective, we can conclude that φ is an isomorphism. Therefore,

$$G \cong H.$$

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3. (13.5) Let G be the group of all real-valued functions on the real line, under addition of functions. Let H be the subset of G consisting of all f such that $f(0) = 0$.

   (a) Show that $H \lhd G$.

   Solution: In part (b), we will find a surjective homomorphism $\varphi : G \to (\mathbb{R}, +)$ whose kernel is H, then apply the first isomorphism theorem to conclude that $G/H \cong (\mathbb{R}, +)$. In fact, that reasoning also guarantees to us that $H \lhd G$, because all kernels are normal in their domains. So, we don't really need to prove that $H \lhd G$ on its own first.

   Even if we didn't realize this, the proof is easy: G is abelian (because additional is abelian), from which it follows that *all* subgroups are normal. □

   (b) Show that $G/H \cong (\mathbb{R}, +)$.

   Solution: As outlined in part (a), our strategy requires us to find a surjective homomorphism $\varphi : G \to (\mathbb{R}, +)$ whose kernel is H. A little reverse-engineering will prove enlightening. We know that we need to map each function to a real number, and the kernel of the map will be those functions that map to 0. Moreover, we *want* the kernel to contains those functions which take the value 0 at 0.

Now, with this in mind, suppose $f \in G$. What should $\varphi(f)$ be? It must be a real number, and it must be 0 when $f(0) = 0$. So, define $\varphi(f) = f(0)$. (In words, φ takes as input a function f, and gives as output the value of f at 0.)

Is this a homomorphism? Well,

$$\varphi(f+g) = (f+g)(0) = f(0) + g(0) = \varphi(f) + \varphi(g),$$

proving that it is.

Is φ surjective? Let $r \in \mathbb{R}$ be arbitrary. We only need to find a single function $f \in G$ such that $\varphi(f) = r$. Choosing, for example, the constant function c_r which takes value r at all points suffices: $\varphi(c_r) = c_r(0) = r$.

What is the kernel of φ? By the definition of the kernel

$$\begin{aligned}\ker(\varphi) &= \{f \in G : \varphi(f) = 0\} \\ &= \{f \in G : f(0) = 0\} \\ &= H.\end{aligned}$$

Hence, by the first isomorphism theorem, we can conclude that

$$G/\ker(\varphi) \cong (\mathbb{R}, +),$$

and so

$$G/H \cong (\mathbb{R}, +).$$

□

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4. (13.12) Let G be a group, let $K \lhd G$, and let H be a subgroup of G such that $HK = G$ and $H \cap G = \{e\}$. Show that $G/K \cong H$. (Note: HK means the subgroup $\{hk : h \in H, k \in K\}$. One can prove that if H and K are subgroups of G, *and* if K is normal in G, then HK is a subgroup of G.)

Solution: I believe that this question has a typo in the textbook, which I didn't notice until it was pointed out to me on Tuesday. Thankfully, even with the typo it is a true statement, but much more trivial. (Nonetheless, this is still good practice! If your proof is long, then you didn't understand the question well enough to realize it is trivial.) I believe that the author meant "$H \cap K = \{e\}$" rather than "$H \cap G = \{e\}$", which is how I initially read the question.

Let's now prove the statement as written. Let G be a group, let $K \lhd G$, and let $H \leq G$ such that $HK = G$ and $H \cap G = \{e\}$. If H is a subgroup of G such that $H \cap G = \{e\}$, then H is itself trivial, i.e., $H = \{e\}$. Now, the only way that $HK = G$ is if $K = G$.

This leads us to conclude that $G/K = G/G$, which is a rather silly group consisting of one element (a coset): $G/G = \{G\} = \{e_{G/G}\}$. On the other hand, H is also a group of size one: $H = \{e_G\}$. All groups of size 1 are isomorphic (as evidenced, if it's really necessary, by the map that sends $e_{G/G}$ to e_G, which, being on a domain and codomain of one element each, is by necessity a bijection and a homomorphism). Therefore, $G/K \cong H$. □
