NAME : ______________________________

Math 31

Midterm 2
August 27, 2017

Prof. Pantone

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have 180 minutes and you should attempt all problems.

- Print your name in the space provided.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

TIPS:

- You don't have numerically expand all answers.
- Use scratch paper to figure out your answers and proofs before writing them on your exam.
- Work cleanly and neatly; this makes it easier to give partial credit.

Problem	Points	Score
1	24	
2	8	
3	8	
4	8	
5	10	
6	10	
7	8	
8	8	
9	8	
10	8	
11	0	
Total	100	

Section 1: True/False.

1. (24) Choose the correct answer. *No justification is required for your answers. No partial credit will be awarded.*

(a) Let $n \geq 4$. The ideal generated by the element 2 in $(\mathbb{Z}_n, \oplus, \otimes)$ is maximal.

True False

(b) The quaternion group Q_8 has an element of order 8.

True False

(c) Let G be a group and let $H \leq G$. If $[G : H] = 2$ then $H \lhd G$.

True False

(d) Let $\varphi : G \to K$ be a group homomorphism. All normal subgroups of G contain $\ker(\varphi)$.

True False

(e) Up to isomorphism there are exactly five finite abelian groups of order 48.

True False

(f) Every infinite abelian group has at least one element of infinite order.

True False

(g) Every infinite cyclic group has at least one element of infinite order.

True False

(h) A finite group is abelian if and only if all of its subgroups are normal.

True False

(i) An integral domain is a ring in which all non-units are zero-divisors.

True False

(j) S_{10} has an element of order 21.

True False

(k) All prime ideals are maximal.

True False

(l) Every polynomial of degree 2 in $\mathbb{C}[X]$ is reducible.

True False

Section 2: Free Response.

You must show all work to receive credit. If you need more space you may use the back of the page. You must clearly indicate on the front of the page that there is more work on the back of the page. Please work neatly.

2. (8) Let G be a group of order pq, for primes p and q. Show that all proper subgroups of G are cyclic.

3. (8) Which of the four maps below are group homomorphisms? (Either show briefly that they are, or give an example that shows they aren't.)

(a) $G = (\mathbb{Q}, +)$, $\varphi : G \to G$ given by $\varphi(x) = |x|$

(b) $G =$ the group of polynomials with integer coefficients under addition of polynomials, $\varphi : G \to G$ given by $\varphi(p(x)) = p'(x)$ (where $p'(x)$ is the usual derivative of a polynomial from calculus)

(c) $G = (\mathbb{R}^+, \cdot)$, $H = (\mathbb{R}, +)$, $\varphi : G \to H$ given by $\varphi(x) = \log_3(x)$.

(d) $G =$ the group of all 2×2 matrices with real entries under addition of matrices, $H = (\mathbb{R}, +)$, $\varphi : G \to H$ given by $\varphi(M) =$ the product of the entries of M.

4. (8) Let R be a commutative ring with 1 and let $a \in R$. Prove that $aR = R$ if and only if a is a unit. (Recall that aR is the ideal generated by a.)

5. (10) Let $\varphi : G \to K$ be an onto homomorphism. Let $J \lhd K$. Prove that there exists a normal subgroup H of G such that $G/H \cong K/J$.

6. (10) (Be sure to justify your answers.)

(a) Is $\mathbb{Q}[X]/(X^2 - 1)$ an integral domain?

(b) Is $\mathbb{Q}[X]/(X^2 + 1)$ a field?

7. (8) Find the right cosets of the subgroup $H = \{(0,0),(2,0),(0,2),(2,2)\}$ in $\mathbb{Z}_4 \times \mathbb{Z}_4$ (under addition).

8. (8) Show that the group $(\mathbb{Q}^+, \cdot)$ is not cyclic.

9. (8) Find, up to isomorphism, all finite abelian groups of order 600.

10. (8) Let $\varphi : R \to S$ be a ring homomorphism. Show that if J is a prime ideal in S then $\varphi^{-1}(J)$ is a prime ideal in R. (You may assume that $\varphi^{-1}(J)$ is an ideal in R.)

11. (0) **Bonus:** *(3 points)* Tell me a little bit about how you prepared for this class (what were your study techniques, did you cram or spread out the work, etc.) What worked for you and what didn't? This will help me give advice to future classes.