NAME : ________________________________

Math 31

Midterm 2
August 7, 2017

Prof. Pantone

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have 120 minutes and you should attempt all problems.

- Print your name in the space provided.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

TIPS:

- You don't have numerically expand all answers.
- Use scratch paper to figure out your answers and proofs before writing them on your exam.
- Work cleanly and neatly; this makes it easier to give partial credit.

Problem	Points	Score
1	20	
2	20	
3	12	
4	12	
5	12	
6	12	
7	12	
Total	100	

Section 1: True/False.

1. (20) Choose the correct answer. *No justification is required for your answers. No partial credit will be awarded.*

(a) The group $A_3 \times \mathbb{Z}_4$ has an element of order 8.

True False

(b) All right cosets of a subgroup H in a group G have the same size.

True False

(c) Every permutation can be written as the product of transpositions.

True False

(d) If k evenly divides $|G|$, then there exists a subgroup H of G such that $|H| = k$.

True False

(e) If G is an infinite group, then G has subgroups of all positive integer orders.

True False

(f) Let G be a group. Then $|G|$ is prime if and only if G is cyclic.

True False

(g) If G is abelian, then $G/Z(G)$ is cyclic.

True False

(h) If $G/Z(G)$ is cyclic, then G is abelian.

True False

(i) $\mathbb{Z}_4 \times \mathbb{Z}_5 \cong \mathbb{Z}_{20}$

True False

(j) If $\pi \in S_n$, then $o(\pi)$ divides n.

True False

Section 2: Short Response.

2. (20) *Justify all answers unless otherwise stated.*

(a) What are the distinct right cosets of $\{1, -1\}$ in $(\mathbb{R} \setminus \{0\}, \cdot)$? Give a group that is isomorphic to $(\mathbb{R} \setminus \{0\})/\{1, -1\}$. (You do not need to justify the isomorphism.)

(b) Give a cyclic subgroup of order four and a non-cyclic subgroup of order four of S_4.

(c) What is the largest order of an element in S_7? In S_10?

(d) Write the permutation below in: one-line notation, disjoint cycle notation, and as a product of transpositions. What is its order?

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 5 & 2 & 8 & 7 & 6 & 10 & 4 & 9 & 3 \end{pmatrix}$$

Section 3: Free Response.

You must show all work to receive credit. If you need more space you may use the back of the page. You must clearly indicate on the front of the page that there is more work on the back of the page. Please work neatly.

3. (12) Let G be a group of size at least two. Prove that if G has no proper non-trivial subgroups, then $|G|$ is prime.

4. (12) Suppose that G is a group, that $N \triangleleft G$, and that G/N abelian. Prove that for all $a, b \in G$, $aba^{-1}b^{-1} \in N$.

5. (12) Prove that $(\mathbb{Z}, +)$ has infinitely many subgroups that are isomorphic to it.

6. (12) Let $\varphi : G \to H$ be an onto homomorphism. Show that if G is cyclic, so is H.

7. (12) Prove that for any group G, it is not possible that $[G : Z(G)]$ is prime. (*Hint:* Consider $G/Z(G)$.)