NAME : Key

Math 31

Midterm 1
July 12, 2017

Prof. Pantone

INSTRUCTIONS: This is a closed book exam and no notes are allowed. You are not to provide or receive help from any outside source during the exam except that you may ask the instructor for clarification of a problem. You have 120 minutes and you should attempt all problems.

- Print your name in the space provided.
- Calculators or other computing devices are not allowed.
- Except when indicated, you must show all work and give justification for your answer. **A correct answer with incorrect work will be considered wrong.**

All work on this exam should be completed in accordance with the Dartmouth Academic Honor Principle.

TIPS:

- You don't have numerically expand all answers.
- Use scratch paper to figure out your answers and proofs before writing them on your exam.
- Work cleanly and neatly; this makes it easier to give partial credit.

Problem	Points	Score
1	20	
2	20	
3	15	
4	15	
5	15	
6	15	
Total	100	

2pts each

Section 1: True/False.

1. (20) Choose the correct answer. *No justification is required for your answers. No partial credit will be awarded.*

(a) The set of real numbers under addition is a group.

True   False

(b) The set of real numbers under multiplication is a group.

Doesn't have an identity.

True   False

(c) The set of even integers under addition is a group.

True   False

(d) The subset of real numbers $\{r \in \mathbb{R} : r \leq -1 \text{ or } r \geq 1 \text{ or } r = 0\}$ under addition is a group.

Not a valid operation.
Ex: $(1.5)+(-1) = 0.5$, which is not in the set.

True   False

(e) The subset of real numbers $\{r \in \mathbb{R} : -1 \leq r \leq 1\}$ under addition is a group.

Not a valid operation.
Ex: $1+1=2$, which is not in the set

True   False

(f) The direct product of any two cyclic groups is cyclic.

True False

Ex: $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(g) The direct product of any two abelian groups is abelian.

True False

(h) If G is not cyclic, then G has a proper subgroup (i.e., a subgroup that's not all of G) that's not cyclic.

Ex: $\mathbb{Z}_2 \times \mathbb{Z}_2$

True False

(i) If G is cyclic, then all subgroups of G are cyclic.

True False

(j) If G is a finite group of order n and if ℓ evenly divides n, then G has an element of order ℓ.

Ex: $\mathbb{Z}_2 \times \mathbb{Z}_2$ yet again!

True False

Another ex: $(\mathcal{P}(\{1,2,3,4,5\}), \Delta)$ has order 32, but the only possible order of an element is 1 or 2. Not 4, 8, 16, 32.

$|\mathbb{Z}_2 \times \mathbb{Z}_2| = 4$, and 4 evenly divides 4, but $\mathbb{Z}_2 \times \mathbb{Z}_2$ has no element of order 4.

These have many possible answers. (b-e)

Section 2: Fill in the blank.

4pts each

2. (20) *No justification is required for your answers, unless otherwise stated. No partial credit will be awarded.*

(a) Write down the elements of $\mathbb{Z}_{10}$ along with the order of each element. (No justification needed.)

$\mathbb{Z}_{10} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

orders: 1 10 5 10 5 2 5 10 5 10

Theorem 4.4.iii: (rephrased for $\mathbb{Z}_n$) In $\mathbb{Z}_n$, $|x| = \frac{n}{\gcd(x,n)}$.

(b) Give an example of a group G and subgroups H and K such that $H \cup K$ is not a subgroup. (No justification needed.)

$G = (\mathbb{Z}, +)$
$H = 2\mathbb{Z}$
$K = 3\mathbb{Z}$

$5 = 2+3$, where $2 \in 2\mathbb{Z}$, $3 \in 3\mathbb{Z}$

But $5 \notin 2\mathbb{Z} \cup 3\mathbb{Z}$.

(c) Give an example of an infinite group G such that every subgroup except $\{e\}$ is also infinite. (No justification needed.)

$(\mathbb{Z}, +)$, $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$

(d) Give an example of a noncyclic group of order 100 along with a one sentence explanation why your group is noncyclic. (Hint: If G and H are finite, then $|G \times H| = |G| \cdot |H|$.)

$\mathbb{Z}_{10} \times \mathbb{Z}_{10}$

Theorem from class: $G \times H$ cyclic iff G cyclic, H cyclic, and $\gcd(|G|, |H|) = 1$.

But $\gcd(10, 10) = 10 \neq 1$.

(e) ~~Give an example of an infinite nonabelian group G such that $Z(G)$ is finite, but contains more than just the identity. You must state both G and $Z(G)$. Recall that $Z(G)$ means "the center of G". (No justification necessary.)~~

I messed up! I meant to ask for an example where $Z(G)$ is <u>infinite</u> but not all of G. I gave everyone full credit, but here's an answer to that question:

$G = GL(2, \mathbb{R})$

$Z(G) = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : \begin{array}{l} a \neq 0 \\ a \in \mathbb{R} \end{array} \right\}$

(from class)

<u>Note:</u> You can find an example with Monday's material, though. Let G be the group of all bijections from $\mathbb{N}$ to $\mathbb{N}$ under composition. Then, $G \times \mathbb{Z}_2$ works.

Section 3: Free Response.

You must show all work to receive credit. If you need more space you may use the back of the page. You must clearly indicate on the front of the page that there is more work on the back of the page. Please work neatly.

3. (15) Let G be a group and let H and K be subgroups of G. Prove that $H \cap K$ is a subgroup of G.

We will use the theorem from class that says S is a subgroup if it's a subset, nonempty, closed under multiplication, and closed under inverses.

(1) $H \cap K$ is a subset of G because $H \cap K \subseteq H$ and $H \subseteq G$.

(2) $H \cap K$ is nonempty because e (the identity) is in H and K, thus $H \cap K$.

(3) Let $a, b \in H \cap K$. Then $a, b \in H$, so $ab \in H$ b/c H is a subgroup. Similarly, $a, b \in K$, so $ab \in K$. Thus $ab \in H \cap K$.

(4) Let $a \in H \cap K$. Then $a \in H$, and so $a^{-1} \in H$ b/c H is a subgroup. Similarly, $a \in K$, and so $a^{-1} \in K$ b/c K is a subgroup. Thus, $a^{-1} \in H \cap K$.

Therefore, $H \cap K$ is a subgroup of G.

$\blacksquare$

4. (15) Let G be a finite abelian group consisting of the elements $\{a_1, a_2, \ldots, a_n\}$. Define c to be the product of all elements in G: $c = a_1 a_2 \cdots a_n$. Prove that $c^2 = e$, where e is the identity element of G.

<u>Proof</u>

Let $c = a_1 a_2 \cdots a_n$. Then, $c^2 = (a_1 a_2 \cdots a_n)(a_1 a_2 \cdots a_n)$.

The inverse of a_1 is some element of G, so it is one of the a_i. (It might be a_1 itself, if $a_1^{-1} = a_1$, or it might be another one, like a_7.)

The inverse of a_2 is some element of G, so it is one of the a_i, but not the same one as a_1^{-1}, because inverses are unique.

In general, $\{a_1, a_2, \ldots, a_n\}$
and $\{a_1^{-1}, a_2^{-1}, \ldots, a_n^{-1}\}$
are the same sets, just in a different order!

So, $a_1 a_2 \cdots a_n = a_1^{-1} a_2^{-1} \cdots a_n^{-1}$ (because G is abelian)

Thus $c^2 = (a_1 a_2 \cdots a_n)(a_1^{-1} a_2^{-1} \cdots a_n^{-1}) = (a_1 a_1^{-1})(a_2 a_2^{-1}) \cdots (a_n a_n^{-1}) = e.$

<u>Note:</u> See if you can prove that if G is as above <u>and</u> has no elements of order 2, then $c = e$.

5. (15) Let G be a group and let $x, y \in G$. Prove that $o(xy) = o(yx)$.

We are not assuming G is abelian!

Proof:

We really need to show two things:

(I) $o(xy) = \infty$ iff $o(yx) = \infty$

AND

(II). $o(xy) = n$ iff $o(yx) = n$ for all $n \in \mathbb{N}$.

First we'll show (II). Let $o(xy) = N$. Then, by definition,

$$\underbrace{xyxy \cdots xy}_{N \text{ "}xy\text{"s}} = e.$$

Multiplying on the left by y and on the right by y^{-1}:

$$y(xyxy \cdots xy)y^{-1} = yy^{-1} = e$$

$$\parallel$$

$$\underbrace{yxyx \cdots yx}_{N \text{ "}yx\text{"s}}.$$

This proves that $o(yx) \leq N$. (i.e., $o(yx) \leq o(xy)$) We need ~~to~~ also check that $(yx)^i \neq e$ for $1 \leq i \leq N-1$.

But, the same argument works in reverse: if $o(yx) = N$, it shows $o(xy) \leq N$. Hence $o(yx) \leq o(xy)$ and $o(xy) \leq o(yx)$, from which we conclude that $o(xy) = o(yx)$. So, (II) is proved.

The contrapositive of (I) says $o(xy) < \infty$ iff $o(yx) < \infty$, and the above logic verifies this as well. $\blacksquare$

6. (15)

(a) Let G be a finite group. Suppose that $o(x) = |G|$ for all $x \in G$ except for the identity. Prove that G has no subgroups other than itself and the subgroup containing only the identity.

Proof: Suppose that H is a subgroup of G other than $\{e\}$. Then H contains a non identity element $y \in G$, with $o(y) = |G|$. The group $\langle y \rangle$ is a subgroup of H, and $|\langle y \rangle| = |G|$. This only makes sense if $\langle y \rangle = H = G$.

So, we started with an arbitrary non-identity subgroup, and showed it had to equal the whole group. Thus, G has no subgroups other than e and itself. ∎

(b) Prove that a group G satisfying the above criteria must be abelian.

Proof: In the proof of (a), we encountered the fact that $G = \langle y \rangle$, i.e., G is cyclic. Cyclic groups are always abelian (theorem from class.)

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