MATH 20 – HOMEWORK 8

due Wednesday, August 23

Instructions: This assignment is due at the *beginning* of class. Staple your work together (do not just fold over the corner). Please write the questions in the correct order. If I cannot read your handwriting, you won't receive full credit.

You may use Wolfram Alpha or another similar tool to compute any necessary sums or integrals, and for your matrix calculations. If you have trouble with this, let me know.

If you're using facts about distributions to answer the questions, be very clear about which distribution you're using to model that problem and why that distribution is appropriate.

1. Suppose a stranger approaches you with two envelopes containing money, with one envelope containing n times as much as the other, where n is any positive integer greater than one. He gives you one at random and allows you to look inside the envelope. He then asks if you would like to switch. Suppose also that you know the probability distribution $p_a = P(\text{smaller envelope contains } a \text{ dollars})$.

   Find an inequality in terms of p_a and $p_{a/n}$ that determines when you expect to benefit from switching. (Recall that we proved in class that when $n = 2$, we switch when $\frac{p_a}{p_a + p_{a/2}} > \frac{1}{3}$.)

2. Suppose that $10,000$ random digits (between 0 and 9) are chosen uniformly at random. Find (a) exactly using a binomial distribution, and (b) approximately using the Central Limit Theorem and a normal distribution table, the probability that the digit 3 appears not more than 931 times. If you have trouble evaluating the sum in part (a) in Wolfram Alpha, tell me.

3. A random walker starts at 0 on the x-axis and at each time unit moves 1 step to the right or 1 step to the left with probability $1/2$. Estimate the probability that, after 100 steps, the walker is more than 10 steps from the starting position. (Use the Central Limit Theorem and a normal distribution table.)

4. A surveyor is measuring the height of a cliff known to be about 1000 feet. He assumes his instrument is properly calibrated and that his measurement errors are independent, with mean $\mu = 0$ and variance $\sigma^2 = 10$. He plans to take n measurements and form the average. Estimate, using (a) Chebyshev's inequality and (b) the normal approximation, how large n should be if he wants to be 95 percent sure that his average falls within 1 foot of the true value. Now estimate, using (c) Chebyshev's inequality and (d) the normal approximation, how small σ^2 must be if he wants to take only 10 measurements with the same resulting confidence.

5. Assume that a student going to a certain four-year medical school in northern New England has, each year, a probability q of flunking out, a probability r of having to repeat the year, and a probability p of moving on to the next year (in the fourth year, this means graduating).

   (a) Form a transition matrix for this process taking as states $F, 1, 2, 3, 4, G$, where F stands for flunking out, G stands for graduating, and the other states represent the year of study.

   (b) For the case $q = 0.1$, $r = 0.2$, $p = 0.7$, find the amount of time a new student can expect to be in medical school?

   (c) Find the probability that this beginning student will graduate.

6. **Bonus: 5 points. (Not required.)** A process moves on the integers 1, 2, 3, 4, 5. It starts at 1 and, on each successive step, moves to an integer greater than its present position, moving with equal probability to each of the remaining larger integers. State five is an absorbing state. Find the expected number of steps to reach state five. Explain how this question is related to the "Stubborn Candles" experiment in Lab 1, and how you could now find *exactly* the expected number of attempts to blow out 35 candles. (You don't need to actually perform this calculation.)

7. **Bonus: 5 points. (Not required.)** A fair coin is flipped 400 times. Determine the number x such that the probability that the number of heads is between $200 - x$ and $200 + x$ is approximately 0.8. (Use the Central Limit Theorem and a normal distribution table.)