MATH 20 – HOMEWORK 7

due Wednesday, August 16

Instructions: This assignment is due at the *beginning* of class. Staple your work together (do not just fold over the corner). Please write the questions in the correct order. If I cannot read your handwriting, you won't receive full credit.

If you're using facts about distributions to answer the questions, be very clear about which distribution you're using to model that problem and why that distribution is appropriate.

1. Suppose that the height, in inches, of a 25-year old man is a normal random variable with parameters $\mu = 71$ and $\sigma^2 = 6.25$. What percentage of 25-year old men are over 6 feet 2 inches tall? What percentage of men over 6 feet tall are over 6 foot 5 inches?

2. Suppose that X and Y are two independent random variables with density functions

$$f_X(x) = \begin{cases} \frac{3}{2}(x+1)^2, & \text{if } -1 \leq x \leq 0 \\ \frac{3}{2}(x-1)^2, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

   and

$$f_Y(x) = \begin{cases} \frac{1}{2}, & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}.$$

   Find the density function for $Z = X + Y$.

3. In this exercise, you will use the notion of convolution and the principle of mathematical induction (a proof technique) to prove that if $X_1, \ldots, X_n$ are independent and identically distributed exponential random variables with rate λ, then the probability density function of $S_n = X_1 + \cdots + X_n$ (for $n \geq 1$) is

$$f_{S_n}(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{n-1}}{(n-1)!}.$$

   Note that we proved the $n = 2$ case explicitly in class.

   Mathematical Induction: Suppose you have a statement that you want to prove is true for all positive integers $n = 1, 2, 3, \ldots$. One way to do this is induction: first you show it's true for $n = 1$ (this is called the base case), then you show that *if* it's true for some number k *then* it must be true for the next number $k + 1$ (this is called the induction step). If you show these two things, then the statement must be true for all n. (Why? Because you showed it's true for 1 in the base case. Then the induction step shows that if it's true for 1 then it must be true for 2. If it's true for 2, then it must be true for 3, etc.)

   (a) We want to prove that $f_{S_n}(x)$ has the form above for all $n \geq 1$. That means that $n = 1$ is our base case. Why do we already know this is true?

   (b) Next prove the induction step. In other words, *assume* that for some integer $k \geq 1$ we already know that

$$f_{S_k}(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^{k-1}}{(k-1)!}.$$

and prove that

$$f_{S_{k+1}}(x) = \frac{\lambda e^{-\lambda x}(\lambda x)^k}{k!}.$$

Hint: $S_k = S_{k-1} + X_k$.

Parts (a) and (b) together prove the statement by the principle of mathematical induction.

4. What do Chebyshev's Inequality and the Law of Large Numbers say about the probability of getting at least 75 heads when flipping a fair coin 100 times? *Hint:* Improve your bound by using the fact that the binomial distribution is symmetric.

5. Each student's score on a particular calculus final is a random variable with values in the range $[0, 100]$, mean 70, and variance 25.

   (a) Find the best lower bound you can (using only the tools we've learned), for the probability that a particular student's score will fall between 65 and 75.

   (b) If 100 students take the final, find a lower bound for the probability that the class mean will fall between 65 and 75.

6. A share of common stock in the Pilsdorff beer company has price Y_n on the nth business day of the year. (Y_n is a random variable.) Finn observes that the price change $X_n = Y_{n+1} - Y_n$ appears to be a random variable with mean $\mu = 0$ and variance $\sigma^2 = 1/4$. If $Y_1 = 30$, find a lower bound for the following probabilities, under the assumption that the X_n's are mutually independent.

   (a) $P(25 \leq Y_2 \leq 35)$

   (b) $P(25 \leq Y_{11} \leq 35)$

   (c) $P(25 \leq Y_{101} \leq 35)$