MATH 20 – HOMEWORK 5 SOLUTIONS!

due Wednesday, August 2

Instructions: This assignment is due at the *beginning* of class. Staple your work together (do not just fold over the corner). Please write the questions in the correct order. If I cannot read your handwriting, you won't receive full credit.

You may use Wolfram Alpha to compute any necessary sums or integrals.

If you're using facts about distributions to answer the questions, be very clear about which distribution you're using to model that problem and why that distribution is appropriate.

1. When you listen to your "Math Homework" playlist on shuffle on Spotify, you usually hear your favorite song about once every two days. If you then go a whole week without hearing it, how surprised are you? (In other words, what's the probability of this occurring?)

   Solution: This situation is best modeled with a Poisson distribution. When your song is played, that is a success, and there are not a fixed number of trials. The rate of success is given to us as

   `1 play / two days`

   but the question asks about one week time scale. So, we translate the given rate to match the time scale:

   $$\lambda = \texttt{7/2 plays / one week.}$$

   The probability distribution function for the Poisson distribution tells us that the probability of zero successes in one unit of time is

   $$P(X = 0) = \frac{\lambda^0 e^{-\lambda}}{0!} = e^{-7/2} \approx 0.0320.$$

   Hence, the probability that this happens is about 3.2%. Personally, I would only be a little surprised.

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2. On an average 8-hour school day, 1000 people walk into Kemeny Hall. Assume this happens completely randomly[1]. What is the probability that exactly six people enter Kemeny Hall in a ten minute span?

   Solution: This is another situation that is modeled by a Poisson distribution. The successes are the arrivals of people. The given rate is

   `1000 people / 8 hours`

   but the question asks about a 10 minutes time scale. As there are $6 \cdot 8 = 48$ ten-minute spans in 8 hours, we adjust the rate to be

   $$\lambda = \texttt{1000/48 people / 10 minutes} = \texttt{125/6 people / 10 minutes.}$$

[1] Of course, this is a terrible assumption—people are more likely to arrive in the short periods between classes. But let's ignore that for now.

The probability of exactly six arrivals in one unit of time is

$$P(X=6)=\frac{\lambda^6 e^{-\lambda}}{6!}=0.0001017.$$

So, at 0.01%, this even is extremely unlikely.

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3. Let $X_1, X_2, \ldots, X_k$ be k random variables that are mutually independent and uniformly distributed on the interval $[0,1]$. Define a new random variable $Y=\min(X_1, X_2, \ldots, X_k)$ such that the value of Y is the smallest of the values of $X_1, X_2, \ldots, X_k$. Find $\mathbb{E}[Y]$.

Solution: In order to find the expected value of the continuous random variable Y, we must first figure out its probability density function. To do this, we'll instead figure out the cumulative density function, then differentiate. In other words, we must determine

$$P(Y \leq r)$$

for all real numbers r. Because each X_i only takes values on $[0,1]$, it's easy to see that if $r<0$ then $P(Y \leq r)=0$ and if $r>1$ then $P(Y \leq r)=1$. What about $P(Y \leq r)$ for some $r \in [0,1]$?

First, note that for all r, $P(Y \leq r)=1-P(Y>r)$, as the events $Y \leq r$ and $Y>r$ are complements. In order for the minimum of k random variables to be greater than r, all of the random variables themselves must take value greater than r. Phrased mathematically, the event

$$Y>r$$

is equivalent to

$$X_1>r, \qquad X_2>r, \qquad \cdots \qquad X_k>r.$$

Since each X_k is uniformly distributed on $[0,1]$,

$$P(X_i>r)=1-r$$

for all i. As these events are independent,

$$P(Y>r)=P(X_i>r \text{ for all } i)=P(X_1>r)P(X_2>r)\cdots P(X_k>r)=(1-r)^k.$$

This allows us to find the cumulative density function for Y: for $r \in [0,1]$,

$$F(r)=P(Y \leq r)=1-(1-r)^k.$$

The probability density function is found by taking the derivative with respect to r:

$$f(r)=F'(r)=k(1-r)^{k-1},$$

for $r \in [0,1]$ and $f(r)=0$ elsewhere.

Lastly, we compute the expected value:

$$\mathbb{E}[X]=\int_{-\infty}^{\infty} r f(r) dr=\int_0^1 kr(1-r)^{k-1} dr=\frac{1}{k+1}.$$

[The integral can be computed using Wolfram Alpha.]

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4. Let X be a discrete random variable that takes only positive integer values. Our normal formula for the expected value of X says

$$\mathbb{E}[X]=\sum_{k=1}^{\infty} kP(X=k).$$

Prove the following alternate formula:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} P(X \geq k).$$

Solution: The common formula for expected value is

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} kP(X = k).$$

Since k is always an integer, we can use the fact that

$$k \cdot P(X = k) = \underbrace{P(X = k) + P(X = k) + \cdots + P(X = k)}_{k \text{ times}}.$$

Note also that since X takes only positive integer values

$$P(X \geq k) = P(X = k) + P(X = k + 1) + \cdots . \quad (1)$$

Suppose that we take the sum

$$\sum_{k=1}^{\infty} P(X \geq k)$$

and use equation (1) to expand. We get

$$\sum_{k=1}^{\infty} (P(X = k) + P(X = k + 1) + P(X = k + 2) + \cdots).$$

How often is a particular term $P(X = N)$ counted in this sum? It appears once in $P(X \geq 1)$, once in $P(X \geq 2)$, etc, all the way up to $P(X \geq N)$, and then it doesn't appear in the rest of the terms. Thus, $P(X = N)$ is counted N times—once in each of the first N terms. Therefore,

$$\sum_{k=1}^{\infty} P(X \geq k) = \sum_{k=1}^{\infty} kP(X = k) = \mathbb{E}[X]$$

and the identity is proved.