MATH 20 – HOMEWORK 4 SOLUTIONS!

due Wednesday, July 26

Instructions: This assignment is due at the *beginning* of class. Staple your work together (do not just fold over the corner). Please write the questions in the correct order. If I cannot read your handwriting, you won't receive full credit.

1. Give an example where Markov's inequality is actually an equality. That is, give a Ω, m, X, and a such that $P(X \geq a) = \mathbb{E}[X]/a$.

   Solution: We can accomplish this with an incredibly trivial example: a sample space with a single elementary event. Set $\Omega = \{\mathtt{A}\}$, so $m(\mathtt{A}) = 1$, and define X such that $X(\mathtt{A}) = 1$. Then, we can compute directly that $\mathbb{E}[X] = 1$, while on the other hand Markov's inequality says, for $a = 1$, that

   $$P(X \geq 1) \leq 1.$$

   Of course, we know directly that $P(X \geq 1) = 1$, so in this example Markov's inequality is right on the nose.

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2. Prove that for any random variable X,

   $$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

   Solution: By definition, $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2]$. Hence,

   $$\begin{aligned} \text{Var}(X) &= \mathbb{E}[(X - \mathbb{E}[X])^2] \\ &= \mathbb{E}[X^2 - 2\mathbb{E}[X]X + \mathbb{E}[X]^2] \\ &= \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + (\mathbb{E}[X])^2 && \text{(by Linearity of Expectation)} \\ &= \mathbb{E}[X^2] - 2(\mathbb{E}[X])^2 + (\mathbb{E}[X])^2 \\ &= \mathbb{E}[X^2] - (\mathbb{E}[X]^2). \end{aligned}$$

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3. Suppose $\Omega = \{a, b\}$, that $m(a) = m(b) = 1/2$, and that X is a random variable with $X(a) = a$ and $X(b) = b$. Find a formula in terms of a and b for $\mathbb{E}[X]$ and $\text{Var}(X)$.

   Solution: The expected value of X is

   $$\mathbb{E}[X] = \frac{1}{2} \cdot a + \frac{1}{2} \cdot b = \frac{a+b}{2}.$$

The variance is

$$\begin{aligned}\mathrm{Var}(X) &= \frac{1}{2}\cdot\left(a-\frac{a+b}{2}\right)^2+\frac{1}{2}\cdot\left(b-\frac{a+b}{2}\right)^2\\ &= \frac{1}{2}\left(\frac{a-b}{2}\right)^2+\frac{1}{2}\left(\frac{b-a}{2}\right)^2\\ &= \frac{(a-b)^2+(b-a)^2}{8}\\ &= \frac{(a-b)^2}{4}.\end{aligned}$$

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4. Consider flipping a weighted coin with the property that the coin comes up heads with probability p and tails with probability $1-p$. Suppose that you flip the coin repeatedly until it comes up tails. Let X be the random variable for the number of flips completed. Find $\mathbb{E}[X]$ and $\mathrm{Var}(X)$. You may use Wolfram Alpha or another tool to compute the value of any infinite sums.

   Solution: Let's compute a few specific probabilities before we look for a general form. The probability that you flip the coin only once is

   $$P(X=1)=1-p.$$

   The probability you flip it twice is

   $$P(X=2)=p(1-p),$$

   corresponding to the sequence `HT`. The probability you flip it three times is

   $$P(X=3)=p^2(1-p),$$

   corresponding to the sequence `HHT`. In general,

   $$P(X=k)=p^{k-1}(1-p).$$

   We can now compute the expected value.

   $$\mathbb{E}[X]=\sum_{\omega\in\Omega}P(\omega)X(\omega)=\sum_{k=1}^{\infty}P(X=k)\cdot k=\sum_{k=1}^{\infty}kp^{k-1}(1-p)=\frac{1}{1-p}.$$

   The last equality was computed using Wolfram Alpha with the input

   "`sum from k=1 to infinity of (k * p^(k-1) * (1-p)) assuming |p| < 1`".

   Similarly, the variance is

   $$\begin{aligned}\mathrm{Var}(X) &= \mathbb{E}[(X-\mathbb{E}[X])^2]\\ &= \mathbb{E}\left[\left(X-\frac{1}{1-p}\right)^2\right]\\ &= \sum_{\omega\in\Omega}P(\omega)\cdot\left(X-\frac{1}{1-p}\right)^2(\omega)\\ &= \sum_{k=1}^{\infty}(p^{k-1}(1-p))\cdot\left(k-\frac{1}{1-p}\right)^2\\ &= \frac{p}{(1-p)^2}.\end{aligned}$$

Important Note: As Wolfram Alpha is quick to tell you, these sums only converge if $p < 1$. What if $p = 1$? In this case, our sample space cannot be $\Omega = \{1 \text{ flip}, 2 \text{ flips}, \ldots\}$ because all of these events have probability zero, which would mean the total of all their probabilities is zero, not one. It would have to be something like $\Omega = \{\infty \text{ flips}\}$. But then what is X? A random variable must take real number values, so it's not permissible to say $X(\infty) = \infty$. Therefore the question isn't even well-defined when $p = 1$.

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5. Consider flipping a fair coin. Suppose that you flip the coin repeatedly until *you get a heads then a tails consecutively, in that order*. For example, some flipping sequences are: `HT`, `HHHHHT`, and `TTTTHHHT`. Let X be the random variable for the number of flips completed. Find $\mathbb{E}[X]$ and $\text{Var}(X)$. You may use Wolfram Alpha or another tool to compute the value of any infinite sums.

Solution: This one is trickier, because there is more than one possible flip sequence of each length. What is the probability that $X = k$? For this to happen, the sequence of flips must have the form

$$\underbrace{\texttt{TT}\cdots\texttt{T}}_{m \text{ tails}}\underbrace{\texttt{HH}\cdots\texttt{H}}_{n \text{ heads}}\texttt{T},$$

where $m + n = k - 1$. How many possible sequences are there of this form of length k? The number n of heads must be at least 1 and can be as large as $k - 1$. Moreover, once n is chosen, m is completely determined. Hence there are $k - 1$ such sequences.

As an example, we see that for $k = 5$ the allowed sequences are

`TTTHT`, `TTHHT`, `THHHT`, `HHHHT`.

Therefore,

$$P(X = k) = \frac{k-1}{2^k}.$$

We can now find the expected value:

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} \frac{k-1}{2^k} \cdot k = 4,$$

and the variance

$$\text{Var}(x) = \sum_{k=1}^{\infty} \frac{k-1}{2^k} \cdot (k-4)^2 = 4.$$
