MATH 20 – HOMEWORK 3 SOLUTIONS!

due Wednesday, July 19

Instructions: This assignment is due at the *beginning* of class. Staple your work together (do not just fold over the corner). Please write the questions in the correct order. If I cannot read your handwriting, you won't receive full credit.

1. Suppose you flip a penny and a dime. Let X be the result of flipping the penny where we assign the value of Heads to be 2 and the value of Tails to be 1, and let Y be the result of flipping the dime where we assign the value of Heads to be 4 and Tails to be 3. (So, for example, $X(\texttt{heads}) = 2$.) Find $\mathbb{E}[X]$, $\mathbb{E}[Y]$, $\mathbb{E}[X+Y]$ and $\mathbb{E}[XY]$. Compute $\mathrm{Var}(X+Y)$ and $\mathrm{Var}(XY)$.

   Solutions: We can think of our sample space as having four elements: $\Omega = \{\texttt{HH}, \texttt{HT}, \texttt{TH}, \texttt{TT}\}$, where the first of the two letters represents the result of flipping the penny, and the second represents the result of flipping the dime. We can then make the following table, which contains all of the information necessary for finding the requested quantities.

ω	$m(\omega)$	$X(\omega)$	$Y(\omega)$	$(X+Y)(\omega)$	$(XY)(\omega)$
HH	1/4	2	4	6	8
HT	1/4	2	3	5	6
TH	1/4	1	4	5	4
TT	1/4	1	3	4	3

   Using this table, we can directly calculate that:

$$\mathbb{E}[X] = \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 2 + \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 1 = \frac{3}{2}$$

$$\mathbb{E}[Y] = \frac{1}{4}\cdot 4 + \frac{1}{4}\cdot 3 + \frac{1}{4}\cdot 4 + \frac{1}{4}\cdot 3 = \frac{7}{2}$$

$$\mathbb{E}[X+Y] = \frac{1}{4}\cdot 6 + \frac{1}{4}\cdot 5 + \frac{1}{4}\cdot 5 + \frac{1}{4}\cdot 4 = 5$$

$$\mathbb{E}[XY] = \frac{1}{4}\cdot 8 + \frac{1}{4}\cdot 6 + \frac{1}{4}\cdot 4 + \frac{1}{4}\cdot 3 = \frac{21}{4}$$

   Notice that we never used a theorem, just the definition of expected value. We can now observe that $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$ as guaranteed by linearity of expectation, and $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ as given by a theorem about independent random variables.

   To calculate the variances, we just use the definition:

$$\mathrm{Var}(X+Y) = \mathbb{E}[((X+Y) - \mathbb{E}[X+Y])^2].$$

   The inner quantity $((X+Y) - \mathbb{E}[X+Y])^2$ is just a new random variable. We can extend the above table to find its values.

ω	$m(\omega)$	$X(\omega)$	$Y(\omega)$	$(X+Y)(\omega)$	$(XY)(\omega)$	$(X+Y-\mathbb{E}[X+Y])^2$	$(XY-\mathbb{E}[XY])^2$
HH	1/4	2	4	6	8	1	121/16
HT	1/4	2	3	5	6	0	9/16
TH	1/4	1	4	5	4	0	25/16
TT	1/4	1	3	4	3	1	81/16

Hence,

$$\text{Var}(X+Y) = \mathbb{E}[(X+Y-\mathbb{E}[X+Y])^2] = \frac{1}{4}\cdot 1 + \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot 1 = 1/2$$

$$\text{Var}(XY) = \mathbb{E}[(XY-\mathbb{E}[XY])^2] = \frac{1}{4}\cdot\frac{121}{16} + \frac{1}{4}\cdot\frac{9}{16} + \frac{1}{4}\cdot\frac{25}{16} + \frac{1}{4}\cdot\frac{81}{16} = \frac{59}{16}.$$

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2. Repeat the previous question under the assumption that (through some magical forces), the coins always land on the same side (both heads or both tails). For each answer, compare with the one you found in the previous question: are they bigger, smaller, the same? Explain why this is intuitively true.

Solution: To answer this question we can use the exact same table, simply editing the $m(\omega)$ column to reflect the fact that HT and TH are no longer possible.

ω	$m(\omega)$	$X(\omega)$	$Y(\omega)$	$(X+Y)(\omega)$	$(XY)(\omega)$
HH	1/2	2	4	6	8
HT	0	2	3	5	6
TH	0	1	4	5	4
TT	1/2	1	3	4	3

We then find that

$$\mathbb{E}[X] = \frac{3}{2}$$

$$\mathbb{E}[Y] = \frac{7}{2}$$

$$\mathbb{E}[X+Y] = 5$$

$$\mathbb{E}[XY] = \frac{11}{2}.$$

Now we add the two necessary columns to calculate variances. Only one of them changes.

ω	$m(\omega)$	$X(\omega)$	$Y(\omega)$	$(X+Y)(\omega)$	$(XY)(\omega)$	$(X+Y-\mathbb{E}[X+Y])^2$	$(XY-\mathbb{E}[XY])^2$
HH	1/2	2	4	6	8	1	25/4
HT	0	2	3	5	6	0	1/4
TH	0	1	4	5	4	0	9/4
TT	1/2	1	3	4	3	1	25/4

Thus,

$$\text{Var}(X+Y) = 1$$

$$\text{Var}(XY) = \frac{25}{4}.$$

Now let's discuss intuition. The difference between questions 1 and 2 is that in 2 we have imposed a condition that causes X and Y to be dependent. This has no effect on $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ individually, which is why they don't change. By linearity of expectation, if $\mathbb{E}[X]$ and $\mathbb{E}[Y]$ don't change then $\mathbb{E}[X+Y]$ won't change (regardless of dependence).

It's hard to form an intuition about $\mathbb{E}[XY]$. In question 1, $\mathbb{E}[XY] = 21/4 = 5.25$. In question two, the same quantity takes into account only the values 8 and 3, omitting 6 and 4. So we're omitting a value that is 0.75 above the expectation and a value that is 1.25 below the expectation. This means we can expect the new value of $\mathbb{E}[XY]$ to be larger.

Given that $\mathbb{E}[X+Y]$ doesn't change, but we omit two 0 values from the calculation, $\text{Var}(X+Y)$ must increase. Similarly, in the column for $XY - \mathbb{E}[XY]$ the values being omitted are the two smallest, and so it is not surprising that the overall variance increases from $59/16 = 3.6875$ to $25/4 = 6.25$.

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3. Suppose you flip a weighted coin that lands on heads with probability p and tails with probability $1-p$. If we let X be the random variable that assign the value of 1 to Heads and the value of 2 to Tails, then what is $\mathbb{E}[X]$? What is $\text{Var}(X)$.

   Solution: Let's make a table like in the previous question, with $\Omega = \{\mathtt{H}, \mathtt{T}\}$.

ω	$m(\omega)$	$X(\omega)$
H	p	1
T	$1-p$	2

   So, $\mathbb{E}[X] = p \cdot 1 + (1-p) \cdot 2 = p + 2 - 2p = 2 - p$. Now we can add the $(X - \mathbb{E}[X])^2$ column to our table.

ω	$m(\omega)$	$X(\omega)$	$(X - \mathbb{E}[X])^2$
H	p	1	$(1-(2-p))^2 = (1-p)^2$
T	$1-p$	2	$(2-(2-p))^2 = p^2$

   Therefore,

$$\text{Var}(X) = p \cdot (1-p)^2 + (1-p) \cdot p^2 = p(1-p).$$

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4. Someone offers you the following game: You pay \$1 to play the game. They will shuffle a standard 52-card deck and you will choose a card. If it's a face card, you win \$3 (for a total profit of \$2). Otherwise, you win \$0 (for a total profit of -\$1). Assume for the purposes of this question that the face cards are Jack, Queen, and King, and Ace. Phrase this setup in terms of a sample space Ω and with a random variable X describing your profit. What is $\mathbb{E}[X]$? What is $\text{Var}(X)$? Should you play this game?

   Solution: There are several ways to construct a workable Ω. One is to say Ω is the set of the 52 different cards, each with $m(\omega) = 1/52$. Another, simpler option is $\Omega = \{\mathtt{F}, \mathtt{N}\}$ where F represents a face card and N represents a non-face card, accompanied with the probability distribution function $m(\mathtt{F}) = 16/52$ and $m(\mathtt{N}) = 36/52$. Again we make a table for the random variable X, where $X(\mathtt{F}) = 2$ and $X(\mathtt{N}) = -1$.

ω	$m(\omega)$	$X(\omega)$
F	$16/52$	2
N	$36/52$	-1

   We see that

$$\mathbb{E}[X] = \frac{16}{52} \cdot 2 + \frac{36}{52} \cdot (-1) = -\frac{1}{13}.$$

Now we add the column for $(X - \mathbb{E}[X])^2$:

ω	$m(\omega)$	$X(\omega)$	$(X - \mathbb{E}[X])^2(\omega)$
F	16/52	2	729/169
N	36/52	−1	144/169

and we find that

$$\mathrm{Var}(X) = \frac{16}{52} \cdot \frac{729}{169} + \frac{36}{52} \cdot \frac{144}{169} = \frac{324}{169} \approx 1.92.$$

Should you play this game? In real life, that depends on a number of factors. If you expect to lose money, and not that much, and you really enjoy gambling, then maybe it's worth it to play the game. Also a game with high variance is riskier than a game with low variance. For now, just looking at the expected value, I'd say that I would not play. In the long run, I would lose money. (Any answer to this part that makes a reasonable-sounding argument is fine.)

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5. Someone offers you the following game: You roll a fair six-sided die. If you roll a 1, you win \$25. If you roll a 2, you win \$5. If you roll a 3, you win nothing. If you roll a 4 or a 5, you lose \$10. If you roll a 6, you lose \$15. Should you play this game?

Solution: With the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$ we make another table. The random variable X will represent the amount of money we make after a round of the game.

ω	$m(\omega)$	$X(\omega)$
1	1/6	25
2	1/6	5
3	1/6	0
4	1/6	−10
5	1/6	−10
6	1/6	−15

So,

$$\mathbb{E}[X] = \frac{1}{6}(25 + 5 + 0 - 10 - 10 - 15) = -\frac{5}{6}.$$

So, since you expect to lose money in the long run, you should not play this game.
