MATH 20 – HOMEWORK 2 SOLUTIONS!

due Wednesday, July 12

Instructions: This assignment is due at the *beginning* of class. Staple your work together (do not just fold over the corner). Please write the questions in the correct order. If I cannot read your handwriting, you won't receive full credit.

1. *(4 points)* Two archers, Mary and Paul, are shooting at the same target. Mary hits the target 75% of the time and Paul hits the target 25% of the time. Now suppose that both archers shoot one arrow at the target at the same time. If exactly one arrow hits the target, what is the probability that it was shot by Mary?

   Solutions: Let A be the event "Mary hits the target." and let B be the event "Paul hits the target." The events are, by the statement of the question, independent. (This was supposed to be understood, but I clarified this via email.) This allows us to calculate the four probabilities

$$P(A \cap B) = P(A)P(B) = 3/16$$
$$P(\overline{A} \cap B) = P(\overline{A})P(B) = 1/16$$
$$P(A \cap \overline{B}) = P(A)P(\overline{B}) = 9/16$$
$$P(\overline{A} \cap \overline{B}) = P(\overline{A})P(\overline{B}) = 3/16.$$

   Let C be the event that only one person hits the target. Observe that $C = (\overline{A} \cap B) \cup (A \cap \overline{B})$, and so, since $\overline{A} \cap B$ and $A \cap \overline{B}$ are mutually exclusive,

$$P(C) = P(\overline{A} \cap B) + P(A \cap \overline{B}) = 10/16.$$

   The quantity we want to find is $P(A|C)$. By the formula for conditional probability,

$$P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{P(A \cap \overline{B})}{P(C)} = \frac{9/16}{10/16} = \frac{9}{10}.$$

   So, the probability that it was shot by Mary is 90%. In the above line, we used the fact that

$$A \cap C = A \cap ((\overline{A} \cap B) \cup (A \cap \overline{B})) = (A \cap \overline{A} \cap B) \cup (A \cap A \cap \overline{B}) = \emptyset \cup (A \cap \overline{B}) = A \cap \overline{B}.$$

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2. *(4 points)* Imagine that you have three six-sided dice in a bag. Two are normal, fair, six-sided dice, but the third has the numbers 1, 1, 3, 3, 5, 5 on it. (Still, each of the six sides comes up with equal probability.)

   Now suppose that you randomly pick one of the dice out of the bag, and roll it three times (without peeking to see which of the dice it is). If the three rolls come up two 1s and a 3, in any order, then what is the probability that the die you picked is the one labeled 1, 1, 3, 3, 5, 5?

   Solution: Let A be the event "you picked the abnormal die", and let B be the event "you roll $1, 1, 3$" in any order. We want to find $P(A|B)$. By Bayes' Theorem,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\overline{A})P(\overline{A})}.$$

We can compute each of the parts on the right-hand side. For starters, $P(A) = 1/3$ and $P(\overline{A}) = 2/3$. Now we compute the probability that you roll $1, 1, 3$ in any order given that you have the abnormal die. The number of ways to roll $1, 1, 3$ in that order is $2^3 = 8$ (because there are two of each face). The total number of ways to roll three numbers if $6^3 = 216$, so the probability of rolling $1, 1, 3$ in that order is $8/216 = 1/27$. There are two other orders, each with the same probability, yielding a total probability of rolling $1, 1, 3$ (in any order) of $3/27 = 1/9$.

Thus, $P(B|A) = 1/9$. By a similar analysis, $P(B|\overline{A}) = 3/216$. Combining all this information:

$$P(A|B) = \frac{(1/9)(1/3)}{(1/9)(1/3) + (3/216)(2/3)} = 4/5 = 80\%.$$

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3. *(4 points)* In order to test whether a batch of 100 widgets meets specifications, a manufacturer picks 20 items at random. If none are defective, the batch is accepted. If at least one is defective, the manufacturer — whose turns out to be quite unscrupulous — mixes the 20 widgets back in with the 100, then picks 20 more at random. If none are defective, the batch is accepted. If at least one is defective, then the manufacturer finally rejects the batch. If a batch of 100 items has 10 defective widgets, what is the probability that it gets accepted?

   Solution: Let p be the probability that a randomly selected group of 20 items has no defective items. The number of ways to select a group of 20 non-defective items is $\binom{90}{20}$ while the total number of ways to select a group of 20 items is $\binom{100}{20}$. So,

$$p = \frac{\binom{90}{20}}{\binom{100}{20}}.$$

   Let A be the event that first batch has no defective items and let B be the event that the second batch (if necessary) has no defective items. Then, the probability of acceptance is $P(A) + P(\overline{A})P(B|\overline{A})$. As the protocol for the second check is the same as the first check, we conclude that $P(B|\overline{A}) = p$ as well.

   Thus, the probability of acceptance is

$$P(A) + P(\overline{A})P(B|\overline{A}) = p + p(1-p) = 2p - p^2 = 2\frac{\binom{90}{20}}{\binom{100}{20}} - \left(\frac{\binom{90}{20}}{\binom{100}{20}}\right)^2 \approx 18.12\%.$$

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4. *(4 point)* Sue claims that she can distinguish between Pepsi and Coke 75 percent of the time. Mel bets that she cannot and is just guessing randomly. To settle this a bet is made: Sue is to be given ten small glasses, each having been filled with Pepsi or Coke, chosen by tossing a fair coin. Sue wins the bet if she gets seven or more correct. Find the probability that Mel wins if Sue has the ability that she claims. Find the probability that Mel wins if Sue is guessing.

   Solution: First we'll find the probability that Mel wins if Sue is just guessing. Define the events $A_1, \ldots, A_{10}$ such that A_i is the event that Sue guesses correctly on the ith cup. Based on the description of the experiment, it should be clear that $P(A_j|A_i) = P(A_j)$ (in words, the probability that Sue guesses correctly on the jth cup does not change based on whether she guessed correctly on the ith cup). Even stronger, $P(A_j|B) = P(A_j)$, where B is any combination of intersections of any A_i (except A_j) — for example, $P(A_3|A_1 \cap A_2 \cap A_7 \cap A_{10}) = P(A_3)$. So, the 10 events are *mutually independent*, which means that every pair is independent, every group of three is independent, etc.

Let's look at an example of a right/wrong sequence where Sue gets 7 right: `R,W,R,R,R,W,R,W,R,R`. The probability of this particular event occurring is $1/2^{10}$, because each individual guess has $1/2$ probability of occurring, and the events are mutually independent. Therefore, the probability that Sue gets exactly 7 right is

[# of `R/W` sequences where there are 7 `Rs` and 3 `Ws`] · [probability of any particular sequence].

Therefore, the probability that she gets exactly 7 right is

$$\binom{10}{7} \cdot \frac{1}{2^{10}}.$$

The probability that she gets exactly 8 right is

$$\binom{10}{8} \cdot \frac{1}{2^{10}},$$

etc.

So, the probability that she gets at least 7 right is

$$\frac{1}{2^{10}} \left(\binom{10}{7} + \binom{10}{8} + \binom{10}{9} + \binom{10}{10} \right).$$

It is perfectly acceptable to leave your answer in this format, but I'll provide the numerical answer for comparison. With a calculator, we see that the probability that Sue gets at least 7 right is

$$\frac{11}{64} = 17.1875\%.$$

(Conclusion: if Mel wants to be very sure that Sue isn't just guessing, she should probably increase the number of trials!)

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Now we calculate the probability that Sue gets at least 7 right if she really can correctly identify the soda 75% of the time. If A_i is the event that Sue guesses the soda in the ith cup correctly, then in the previous part we have $P(A_i) = 1/2$ and $P(\overline{A_i}) = 1/2$. In this part, however, $P(A_i) = 3/4$ and $P(\overline{A_i}) = 1/4$.

This leads to the following difference: the probability of a specific `R/W` sequence with k `Rs` and ℓ `Ws` is

$$\left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^\ell.$$

Hence, the probability that Sue gets exactly 7 right is

$$\binom{10}{7} \left(\frac{3}{4}\right)^7 \left(\frac{1}{4}\right)^3.$$

Along the same lines, the probably that Sue gets at least 7 right is

$$\binom{10}{7} \left(\frac{3}{4}\right)^7 \left(\frac{1}{4}\right)^3 + \binom{10}{8} \left(\frac{3}{4}\right)^8 \left(\frac{1}{4}\right)^2 + \binom{10}{9} \left(\frac{3}{4}\right)^9 \left(\frac{1}{4}\right)^1 + \binom{10}{10} \left(\frac{3}{4}\right)^{10}.$$

With a calculator, we find that this value is

$$\frac{203391}{262144} \approx 77.59\%.$$

5. *(4 point)* Determine if the following pairs of events are dependent or independent. (You must prove your answer, not just rely on intuition.)

   (a) $A =$ flipping tails on a fair coin, $B =$ rolling a 3 on a fair six-sided die.

   (b) $A =$ drawing a 7 from a deck of cards, $B =$ drawing a heart from a deck of cards

   (c) $A =$ drawing a 3 from a deck of cards, $B =$ drawing an ace from a deck of cards

   Solution: By definition, two events A and B are independent if $P(A \cap B) = P(A)P(B)$.

   In part (a), we imagine that we are simultaneously flipping a coin and rolling a die. The event A describes "the coin comes up tails, and the die comes up anything", while the event B describes "the die comes up 3, the coin comes up anything". To find the probability of A, we note that there are six combinations of ([side of coin], [side of die]) that satisfy the conditions, out of 12 total configurations. Hence, $P(A) = 2/12 = 1/6$. Similarly, $P(B) = 6/12 = 1/2$. Lastly, the $A \cap B$ describes "the coin comes up tails and the die comes up 3". There is only one combination that satisfies this, and therefore $P(A \cap B) = 1/12$. Now we check: is it true that $P(A \cap B) = P(A)P(B)$? Yes. Therefore, these events are independent.

   Moving along to part (b), we are drawing a single card from a deck. The probability that it is a 7 is $4/52 = 1/13$ because there are four 7s and 52 total cards. The probability that it is a heart is $13/52 = 1/4$ because there are thirteen hearts in a deck. The probability that it is a 7 and a heart is $1/52$ because there is only one 7 of hearts in the deck. This translates to the probabilities $P(A) = 1/13$, $P(B) = 1/4$, $P(A \cap B) = 1/52$. Is it true that $P(A \cap B) = P(A)P(B)$? Yes. Therefore, these events are independent.

   In part (c), we use the same logic to deduce that $P(A) = 1/13$ and $P(B) = 1/13$. However $P(A \cap B) = 0$ because it is not possible for a drawn card to be both a 3 and an ace. So this time, $P(A)P(B) \neq P(A \cap B)$ and therefore A and B are dependent.