MATH 20 – HOMEWORK 1 SOLUTIONS!

due Wednesday, July 5

Instructions: This assignment is due at the *beginning* of class. Staple your work together (do not just fold over the corner). Please write the questions in the correct order. If I cannot read your handwriting, you won't receive full credit.

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1. *(3 points)* In how many different ways can the letters of the word `BOOKKEEPER` be rearranged? (For example, the letters in the word `EYE` can be rearranged in three ways: `EEY`, `EYE`, and `YEE`.)

   Solution: The word `BOOKKEEPER` has length 10. If all the letters of the word were different, there would be 10! rearrangements. However, there are two `O`s, and so we need to "forget" the order of the `O`s by dividing by 2!. Similarly, there are two `K`s and three `E`s. The final answer is

   $$\frac{10!}{2!2!3!} = 151200.$$

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2. *(3 points)* Suppose that you draw a seven-card hand at random from a standard deck of 52 cards. What is the probability that your hand contains three of one card and four of another (for example: `3,3,3,3,Q,Q,Q` or `6,6,6,9,9,9,9`)?

   Solution: First we'll count the number of seven-card pokers hands that meet this criterion. To choose such a hand, you have to specify

   (i) which value of card to have four of,
   (ii) which value of card to have three of, and
   (iii) which suit is missing from the value you have three of.

   There are 13 possibilities for the value of the card you have four of, 12 possibilities for the value of the card you have three of, and then 4 possibilities for the missing suit from the cards you have three of.

   Since there are a total of $\binom{52}{7}$ possible seven-card poker hands, the probability of drawing such a hand is

   $$\frac{13 \cdot 12 \cdot 4}{\binom{52}{7}} \approx 0.00046\%.$$

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3. *(3 points)* Suppose that A and B are events. If $P(A \cup B) = 5/7$, $P(\overline{B}) = 4/5$, and $P(A) = 3/5$, then what is $P(A \cap B)$?

   Solution: We will use the formula for class that describes that probability of a union of events:

   $$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

   Solving for $P(A \cap B)$ gives

   $$P(A \cap B) = P(A) + P(B) - P(A \cup B).$$

We're given that $P(A) = 3/5$ and $P(A \cup B) = 5/7$. Moreover, since $P(\overline{B}) = 4/5$ it follows that $P(B) = 1 - 4/5 = 1/5$.

Hence,

$$P(A \cap B) = \frac{3}{5} + \frac{1}{5} - \frac{5}{7} = \frac{3}{35}.$$

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4. *(3 points)* Prove that if A and B are sets, then

$$\overline{A \cup B} = \overline{A} \cap \overline{B}$$

and

$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$

Hint: One way to prove that two sets S and T are equal is to first prove that $S \subseteq T$ and then prove that $T \subseteq S$. The only way both of these are true is if $S = T$.

Note: There are two ways to prove the second theorem: a short application of the first theorem, or the long way mimicking the first theorem.

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Theorem 1: If A and B are sets, then $\overline{A \cup B} = \overline{A} \cap \overline{B}$.

Proof: We will show that $\overline{A \cup B} = \overline{A} \cap \overline{B}$ by first showing that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$ and then showing that $\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$.

First, let $x \in \overline{A \cup B}$ be arbitrary. This implies that $x \notin A \cup B$. For an element to not be in an union, it must be not in both of the sets. We thus conclude that $x \notin A$ and $x \notin B$. By the definition of complement, $x \in \overline{A} \cap \overline{B}$. This proves that $\overline{A \cup B} \subseteq \overline{A} \cap \overline{B}$.

To show the other subset relationship, let $x \in \overline{A} \cap \overline{B}$ be arbitrary. Then $x \in \overline{A}$ and $x \in \overline{B}$, from which it follows that $x \notin A$ and $x \notin B$. Therefore, $x \notin A \cup B$, and we conclude that $x \in \overline{A \cup B}$. Since x was arbitrary, $\overline{A \cup B} \supseteq \overline{A} \cap \overline{B}$.

As we have proved the subset relationship in both directions, we've proved that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. □

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Theorem 2: If A and B are sets, then $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Clever Proof: First note that the complement of the complement of any set is the set itself. So, $\overline{\overline{A}} = A$ and $\overline{\overline{B}} = B$. Hence,

$$\begin{aligned}
\overline{A \cap B} &= \overline{\overline{\overline{A}} \cap \overline{\overline{B}}} \\
&= \overline{\overline{\overline{A} \cup \overline{B}}} && \text{(by applying Theorem 1)} \\
&= \overline{A} \cup \overline{B}.
\end{aligned}$$

This is the end of the proof, but if you have trouble following it, try this rephrased, but completely equivalent, version.

Define $C = \overline{A}$ and $D = \overline{B}$ and observe that $\overline{C} = A$ and $\overline{D} = B$. Now,

$$\overline{A \cap B} = \overline{\overline{C} \cap \overline{D}}. \tag{1}$$

Theorem 1 tells us that $\overline{C} \cap \overline{D} = \overline{C \cup D}$. Substituting this in to equation 1 tells us that

$$\overline{A \cap B} = \overline{\overline{C \cup D}}.$$

Since the complement of the complement of any set is the set itself,

$$\overline{A \cap B} = C \cup D.$$

Substituting the values of C and D back in shows that

$$\overline{A \cap B} = \overline{A} \cup \overline{B}.$$

This completes the proof. □

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Longer Proof: (This proof is very similar to the proof of Theorem 1.) We will show that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ by first showing that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and then showing that $\overline{A \cap B} \supseteq \overline{A} \cup \overline{B}$.

First, let $x \in \overline{A \cap B}$ be arbitrary. This implies that $x \notin A \cap B$. For an element to not be in an intersection, it must be not in at least one of the sets. We thus conclude that either $x \notin A$ or $x \notin B$. By the definition of complement, either $x \in \overline{A}$ or $x \in \overline{B}$, proving that $x \in \overline{A} \cup \overline{B}$. This proves that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.

To show the other subset relationship, let $x \in \overline{A} \cup \overline{B}$ be arbitrary. Then, either $x \in \overline{A}$ or $x \in \overline{B}$, from which it follows that either $x \notin A$ or $x \notin B$. If x is not in at least one of A or B, then $x \notin A \cap B$. Thus, $x \in \overline{A \cap B}$. Since x was arbitrary, $\overline{A \cap B} \supseteq \overline{A} \cup \overline{B}$.

As we have proved the subset relationship in both directions, we've proved that $\overline{A \cap B} = \overline{A} \cup \overline{B}$. □

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5. *(4 points)* Give a proof by contrapositive of the following statement.

If $x + y$ is even then either x and y are both even or x and y are both odd.

Proof by Contrapositive: We will assume that x and y are not both even or both odd, and prove that $x + y$ is not even.

(Note, we have applied DeMorgan's law here to decide how to negate the statement "x and y are both even or x and y are both odd". The negation is "(not [x and y are both even]) and (not [x and y are both odd])".

So, suppose that x and y are not both odd and are not both even. The only remaining possibility is that one is odd, and one is even. Let's assume that x is odd and y is even (the reverse case is identical). So, x has the form $2m + 1$ for some integer m and y has the form $2n$ for some integer n.

Now, $x + y = 2(m + n) + 1$, and therefore $x + y$ is odd and hence not even. □

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6. *(4 points)* Give a proof by contradiction of the following statement.

If n is a positive natural number and n^2 is even, then n is also even.

Proof by Contradiction: To prove by contradiction, we assume that [n is a positive natural number and n^2 is even], and that n is odd. We will derive from this a contradiction.

If n is odd, then we can write it in the form $n = 2k + 1$ for some integer k. Then,

$$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

This shows that n^2 is odd (because it's "two times an integer plus one"), which contradicts our assumption that n^2 is even.

This completes the proof. We assumed that n wasn't even, and found a problem. Therefore n must have been even all along. □
