MATH 4670 / 5670 – COMBINATORICS
HOMEWORK 5

Spring 2024

assigned Wednesday, April 3
due Wednesday, April 17, by the beginning of class

This homework assignment was written in LaTeX. You can find the source code on the course website.

All answers must be fully justified to receive credit. Answers without justification will not be considered correct.

★ Questions that ask you to "prove" something or ask you to "give a proof" should be answered with formal mathematical proofs.

1. (2.4) Find a formula for $P(n, n-4)$ for $n \geq 5$.

2. (2.4) Let $P(n)$ denote the number of integer partitions into any number of parts. Some initial values are

n	1	2	3	4	5	6	7	8	9	10
$P(n)$	1	2	3	5	7	11	15	22	30	42

   Find and prove a formula in terms of $P(n)$ for the number of integer partitions of n that have no parts of size 1.

3. (2.4) In class on Wednesday, March 27, we began to prove a bijection that was going to prove the identity

$$P(n,k) = P(n-k,0) + P(n-k,1) + P(n-k,2) + \cdots + P(n-k,k).$$

   We came up with the function $D : A \to B$ that subtracts 1 from each part and deleting any resulting 0s. Using type vector notation:

$$D([1^{p_1}2^{p_2}\cdots m^{p_m}]) = [1^{p_2}2^{p_3}\cdots(m-1)^{p_m}].$$

   The domain A is integer partitions of n into k parts. The codomain B is integer partitions of $n-k$ into at most k parts. In class we proved this function is well-defined.

   In this exercise, prove that the function is injective. I recommend assuming $p, q \in A$ with $D(p) = D(q)$ and then proving $p = q$. (We started this in class.) Use type vector notation in your proof so that it is clear and precise.

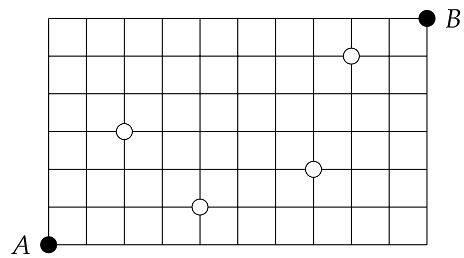
4. (2.4) Prove that the function D in the previous exercise is surjective. Again, use type vector notation and mathematical logic as much as possible.

   Hint: Let $r \in B$, so r is an integer partition of $n-k$ into j parts where $0 \leq j \leq k$. Write r in type vector notation. Find a partition $p \in A$ with the property that $D(p) = r$. Make sure to prove that the p you find really is in A!

5. (3.1) How many functions $[6] \to [7]$ have at most two arrows pointing to each element of the codomain?

6. (3.1) Derive an identity for $\binom{n}{k}$ via inclusion-exclusion by counting the k-multisets of $[n]$ in which each element of $[n]$ appears at most once. Use $p_i =$ "element i appears more than once in the multiset" as the ith property, for $1 \le i \le n$.

7. (3.1) A taxi drives from the intersection labeled A to the intersection labeled B in the grid of streets shown below. The driver only drives north (up) or east (right.)



Traffic reports indicate that there is a heavy congestion at the intersections identified. How many routes from A to B can the driver take that avoid all congested intersections? Your answer should use the idea of inclusion-exclusion.

8. See the graph G in Figure 5.46 on page 102 (in Section 5.9, Exercises) of the free textbook "Applied Combinatorics" by Keller and Trotter. Write G formally as a set V of vertices and E of edges. Then, list the degrees of all of the vertices.

9. Draw a graph with 6 vertices having degrees 5, 4, 4, 2, 1, and 1 or explain why such a graph does not exist.